Orbits in Static and Stationary Spacetimes

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ZARM

Centre for Applied Space Technology and Microgravity (ZARM),
University of Bremen, 28359 Bremen, Germany

1st LARES workshop
Rome, July 3 – 4, 2009

Bremen Drop Tower of ZARM

ZARM = Center for Applied Space Technology and Microgravity

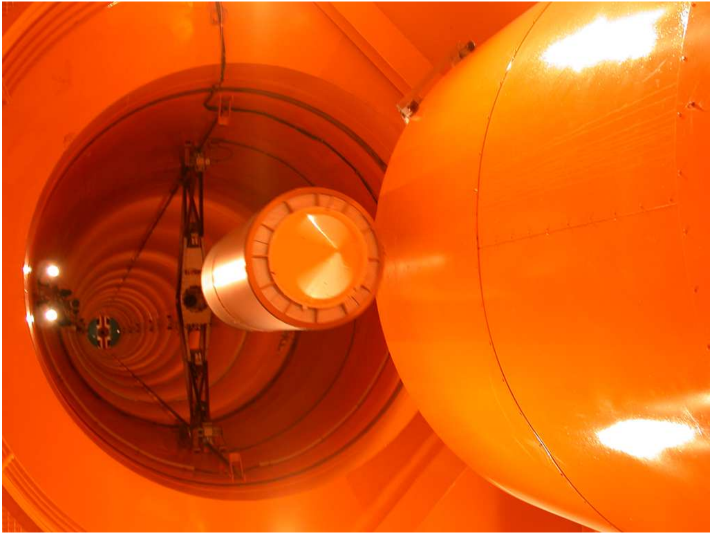
Tower 146 m

drop tube 110 m

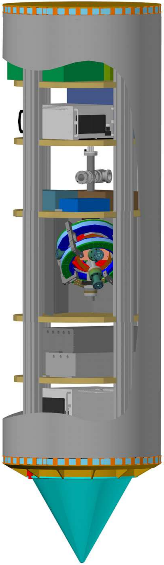
free fall time = 4.7 s

deceleration $\sim 30\ g$

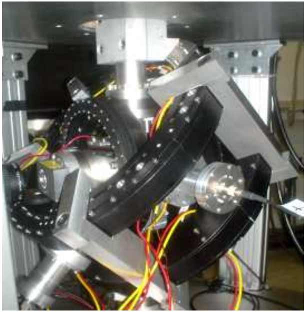
Bremen Drop Tower of ZARM



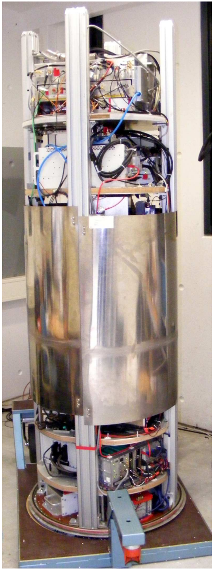
BEC in microgravity



design of capsule



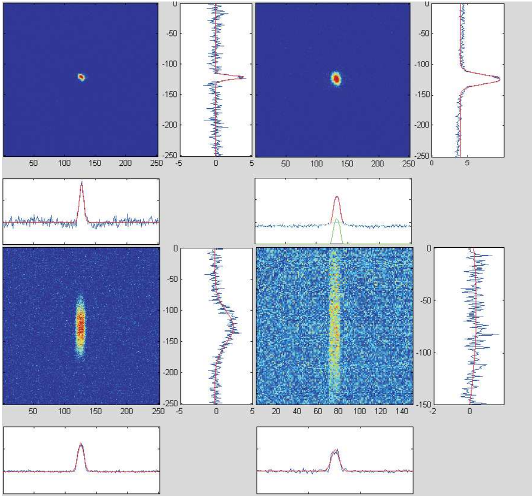
vacuum chamber



capsule

BEC in microgravity – long free evolution

50 ms



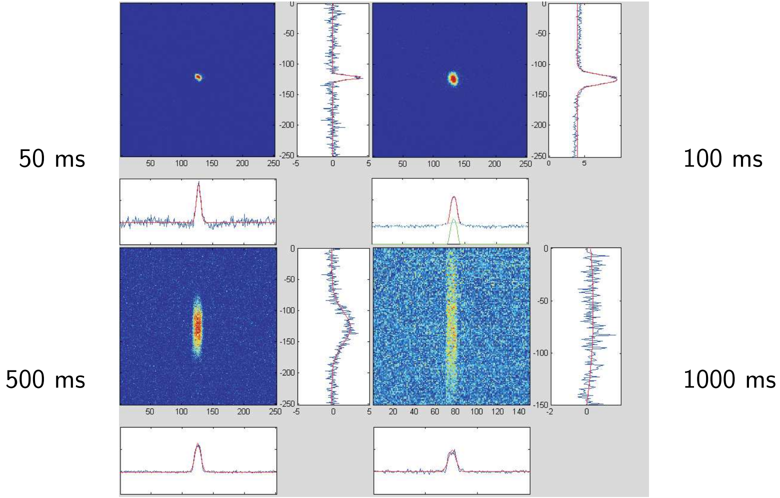
100 ms

500 ms

1000 ms

10^4 atoms, 1 s free evolution time (not possible on ground)

BEC in microgravity – long free evolution



10^4 atoms, 1 s free evolution time (not possible on ground)

Activities at ZARM

- Combustion
- Microfluid dynamics
- Metallic foams
- Fundamental Physics
  - Experiments
    - BEC in free fall (+ theory)
    - Atomic interferometry (+ theory)
    - SQUIDs – test of UFF
  - Modeling
    - Orbit determination
    - Modeling of forces acting on satellites
    - MICROSCOPE
    - Pioneer anomaly
  - Gravity and quantum theory
    - Orbits in space–time
    - Orbits of spinning and extended objects (flyby anomaly)
    - Quantum gravity phenomenology
    - Influence of space–time fluctuations
    - Detection of gravitational waves
    - Questioning/testing Newton's laws
    - Structure of equations of motion
    - Finsler geometry

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Outline

1. Introduction

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2. Motion in gravitational fields
3. Physics in weak gravitational fields

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4. Orbits in Schwarzschild
5. Orbits in Schwarzschild–de Sitter
6. Orbits in higher dimensions
7. Orbits in Kerr
8. Orbits in NUT–de Sitter
9. Orbits in Kerr–de Sitter

Outline

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Introduction

Gravity can only be explored through the motion of test particles

Motion of test particles

- Orbits and clocks
- Massive particles and light

What is gravity or what is the interaction depends on the structure of the equation of motion

- Existence of inertial systems (Finsler)
- Order of differential equation (higher order equations of motion)
- Dependence on particle parameters (Equivalence Principle)

Here we assume validity of standard General Relativity

The present situation

All predictions of General Relativity are experimentally well tested and confirmed

Foundations

The Einstein Equivalence Principle

- Universality of Free Fall
- Universality of Gravitational Redshift
- Local Lorentz Invariance

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Implication

Gravity is a metrical theory

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Implication

Gravity is a metrical theory

⇒

Predictions for metrical theory

- Solar system effects
  - Perihelion shift
  - Gravitational redshift
  - Deflection of light
  - Gravitational time delay
  - Lense–Thirring effect
  - Schiff effect
- "Strong" gravitational fields
  - Binary systems
  - Spin–spin interaction
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General Relativity

$\Leftarrow$

All consequences

Black Holes, Big Bang, ...

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Outline

The framework

The gravitational field

Gravity = space–time metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

What determines the metric

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$$

What determines the motion of particles

Geodesic equation

$$0 = \frac{d^2 x^\mu}{ds} + \left\{ {\mu \atop \rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

There are justifications for that

- Ehlers–Pirani–Schild axiomatic approach
- PPN formalism + experiments/observations

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Metric theory

Predictions

All metric theories imply

- gravitational redshift
- light deflection
- perihelion shift
- gravitational time delay
- Lense–Thirring effect
- Schiff effect
- geodetic precession

Einstein's theory is singled out by certain values for these effects

How to find Einstein's theory? $\longrightarrow$ PPN

Outline

PPN formalism

Physical situation

Spherically symmetric metric $\Rightarrow$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 - g_{rr} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2$$

g_{tt}, g_{rr} $\leftrightarrow$ Gravitational field equations: not known

Parametrization for

- asymptotically flat
- weak fields: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1$

$$\begin{aligned} g_{00} &= -1 + 2\alpha \frac{U}{c^2} - 2\beta \frac{U^2}{c^4}, \qquad U = \text{ Newton potential} \\ g_{0i} &= 0 \\ g_{ij} &= (1 + 2\gamma)\frac{U}{c^2} \end{aligned}$$

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Equations of motion

Space–time

PN metric

$$ds^2 = (1 - 2U)\, dt^2 + 4A_i dx^i dt - (1 + 2U)\, dx^2$$

- U = Newton potential $= \dfrac{M}{r}$
- $\boldsymbol{A}$ = gravitometic vector potential $= \dfrac{\boldsymbol{J} \times \boldsymbol{r}}{r^3}$

PN particle dynamics

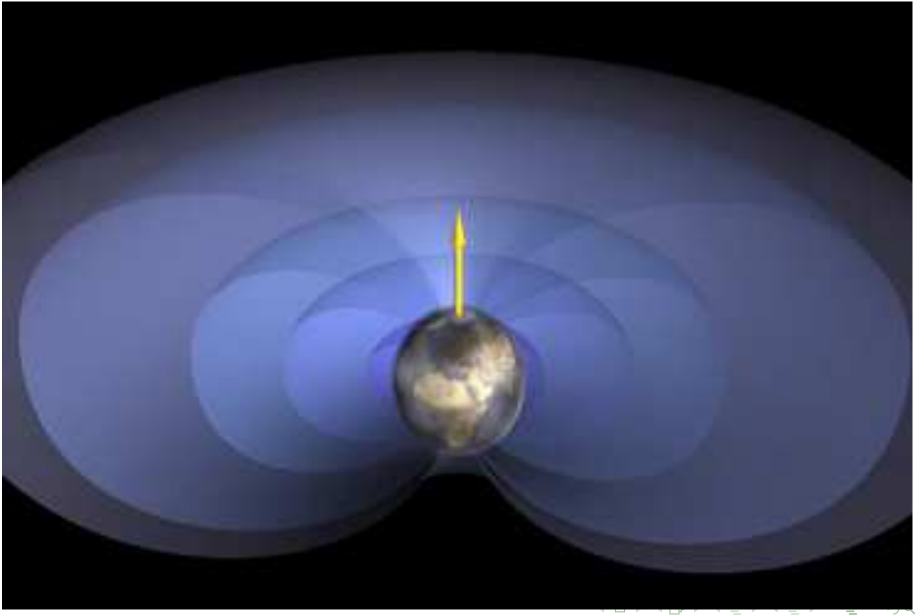
PN Lagrangian

$$L = -m \left(1 - v^2 - 2(1 + v^2)U + 4\boldsymbol{v} \cdot \boldsymbol{A}\right)$$

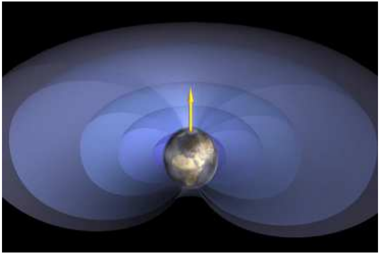
PN equation of motion

$$m\ddot{\boldsymbol{x}} = -m \underbrace{\boldsymbol{\nabla} U}_{\text{gravitoelectric field}} - 2m\boldsymbol{v} \times \underbrace{(\boldsymbol{\nabla} \times \boldsymbol{A})}_{\text{gravitomagnetic field}}$$

The gravitomagnetic fild



The gravitomagnetic field



The gravitomagnetic field

g_{0i}

- Genuine post–Newtonian gravitational field
- Analogue of magnetic field

Main consequences

Notions

- Angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

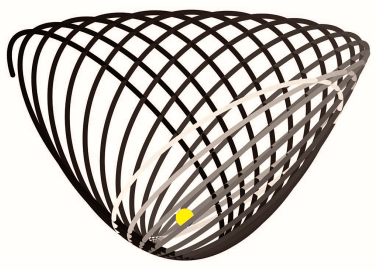
- Runge–Lenz vector

$$\boldsymbol{A} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + Gm\frac{\boldsymbol{r}}{r}$$

Properties

- $\frac{d}{dt}\boldsymbol{L} \neq 0$ (Newton: $= 0$)
- $\frac{d}{dt}\boldsymbol{A} \neq 0$ (Newton: $= 0$)

Lense–Thirring effect



A particular orbit showing the Lense–Thirring effect

Observations

- exact solution: Weierstrass elliptic functions
- Observed quantities

$$\dot{\Omega} = \frac{2GJ}{c^2 a^3 (1 - e^2)^{3/2}}$$
$$\dot{\omega} = -\frac{6GJ\cos i}{c^2 a^3 (1 - e^2)^{3/2}}$$

- Measurement with LAGEOS satellites, together with data from CHAMP and GRACE
- Result: confirmation with approx 10 % error

Gravitomagnetic clock effect

- rotating gravitating mass $\Rightarrow$ Kerr solution

$$ds^2 = \ldots + 4a_\varphi \, d\varphi \, dt + \ldots$$

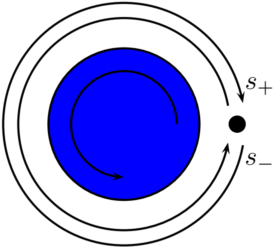
- geodesic equation for circular orbits in equatorial plane

$$\frac{d\varphi}{dt} = \pm\Omega_0 + \Omega_{\mathrm{Lense-Thirring}}$$

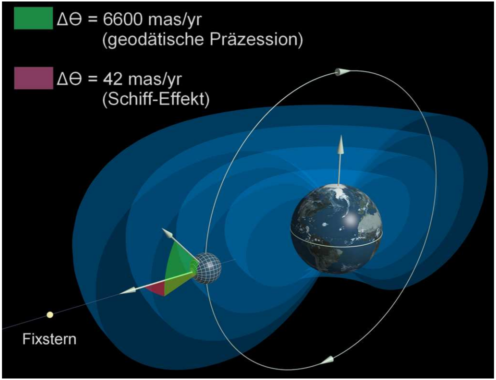
- proper time difference of two counterpropagating clocks

$$s_+ - s_- = 4\pi \frac{J}{M} \sim 10^{-7}\ \mathrm{s}$$

  - is astonishingly large, but requires exact knowledge of orbit
  - does not depend on G and on r
  - decreases with inclination, vanishes for polar orbits



Schiff effect



GP-B

- is different measurement
- can in principle yield different outcome

Description

- Dynamics of direction of spin $D_v S = 0$
- Compared with direction given by distant stars
- Effective dynamics

$$\dot{\boldsymbol{S}} = \boldsymbol{\Omega} \times \boldsymbol{S}$$

with

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{g}$$

- Ongoing data analysis

Outline

Orbits in Schwarzschild space–time

The Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2$$

Equations of motion

Conservation laws

$$E = g_{00} \frac{dt}{ds} \, , \qquad L = r^2 \frac{d\varphi}{ds}$$

With substitutions, e.g., $u = 2M/r$

$$d\frac{\varphi}{2} \quad = \quad \frac{du}{\sqrt{4u^3 - g_2 u - g_3}} = \frac{du}{\sqrt{P_3(u)}}$$

and similar for t and s, with

$$g_2, g_3 \qquad \leftrightarrow \qquad L, E, M$$

Orbits in Schwarzschild space–time

- Solution $u = u(\varphi)$ given through

$$\varphi - \varphi_0 = \int_{u_0}^{u} \frac{du'}{\sqrt{P_3(u')}}$$

- Uniqueness of integration: u is function with 2 periods

$$u(\varphi + 2n\omega_1 + 2m\omega_2) = u(\varphi)$$

with half periods

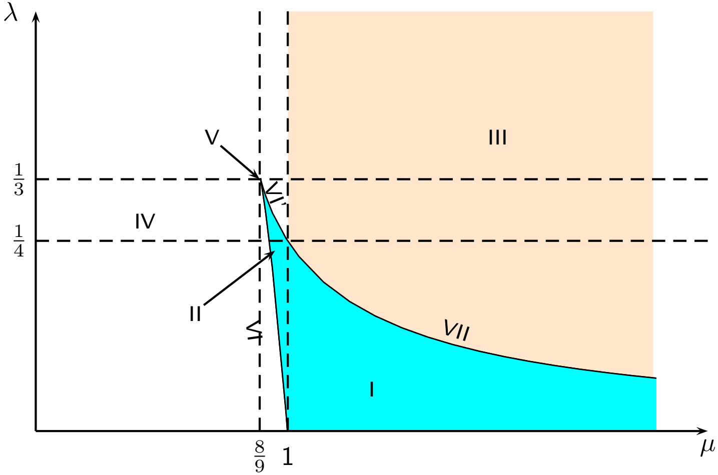
$$\omega_1 = \int_{\mathrm{zero}_1}^{\mathrm{zero}_2} \frac{du}{\sqrt{P_3(u)}}\,, \qquad \omega_2 = \int_{\mathrm{zero}_3}^{\mathrm{zero}_4} \frac{du}{\sqrt{P_3(u)}}$$

Solution for orbit (Hagihara, JJGA 1931)

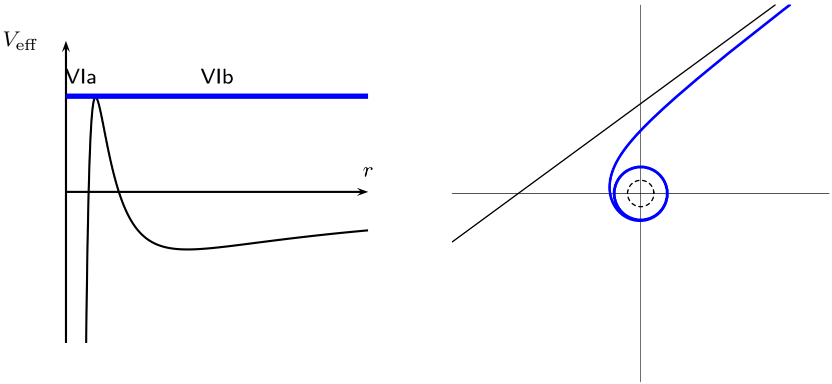
$$r(\varphi) = \frac{2M}{\frac{1}{3} + \wp\left(\frac{\varphi}{2}; g_2, g_3\right)}\,, \qquad r(\varphi) = \frac{2M}{\frac{1}{3} + \wp\left(\frac{\varphi}{2} + i\omega_2; g_2, g_3\right)}$$

Orbits in Schwarzschild space–time

$\lambda - \mu$ parameter plot, discriminant of P_3 (Weierstraß polynomial)

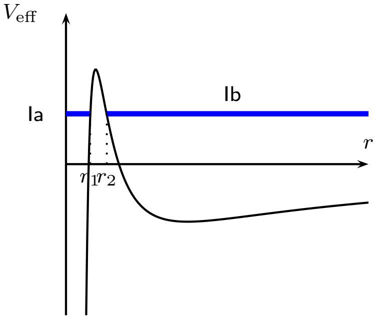


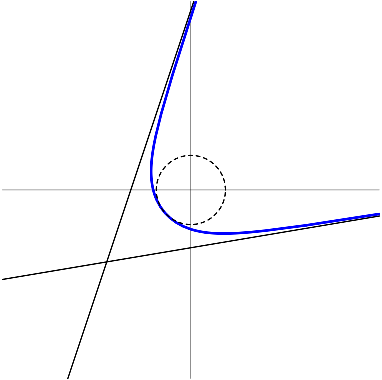
Orbits in Schwarzschild space–time



hyperbolic spiral

Orbits in Schwarzschild space–time

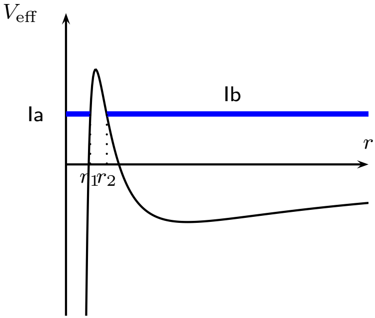


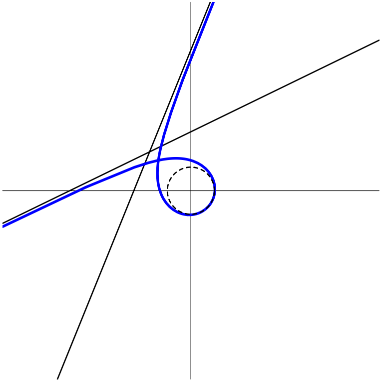


quasi hyperbolic

Deflection angle given by complete elliptic integral as function of $r_{\rm min}$ and E

Orbits in Schwarzschild space–time





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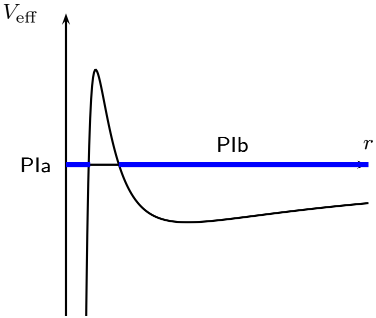

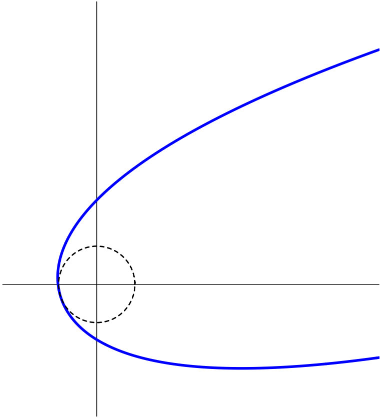

quasi parabolic

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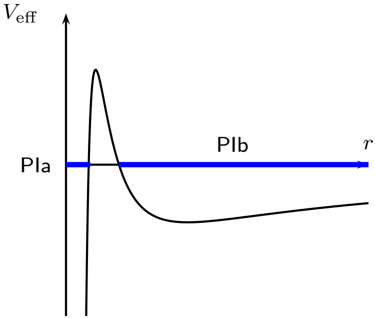

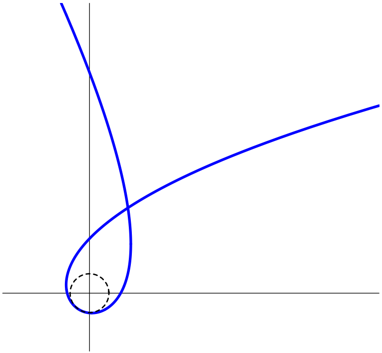

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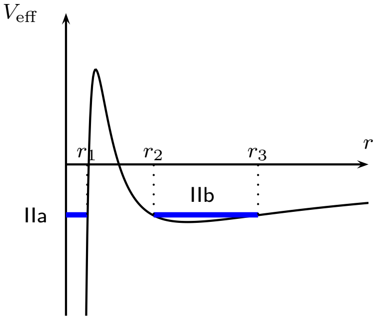
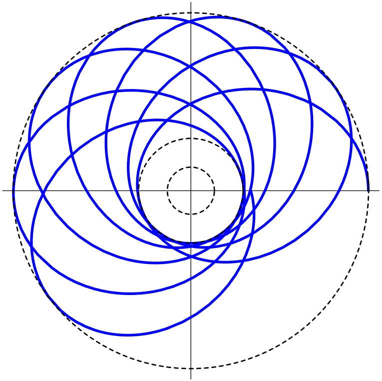
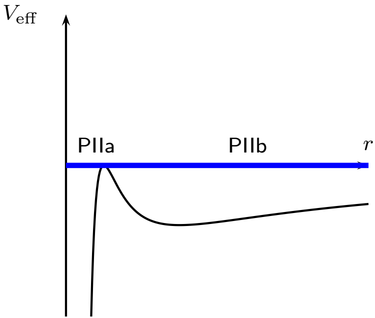


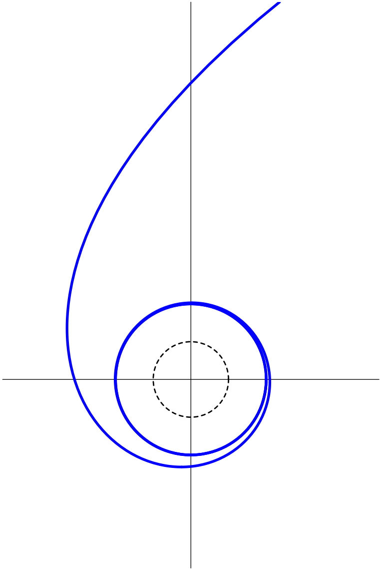


quasi elliptic

perihelion shift: $\delta\varphi = \omega_1 - 2\pi$ analytic function of $r_{\min}$ and $r_{\max}$

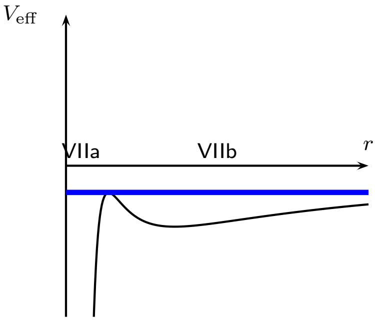
Orbits in Schwarzschild space–time

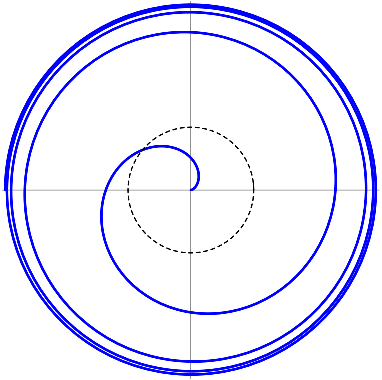




infinite parabolic spiral

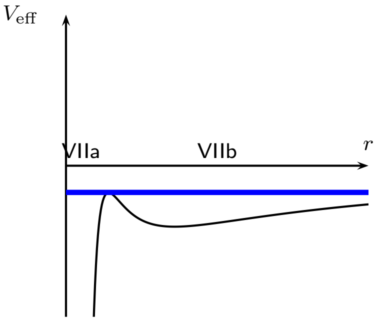
Orbits in Schwarzschild space–time

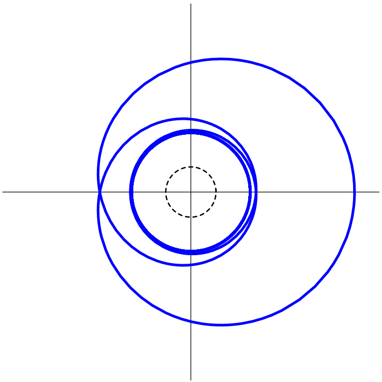




spiral — double spiral (Poincarés double circle limit)

Orbits in Schwarzschild space–time





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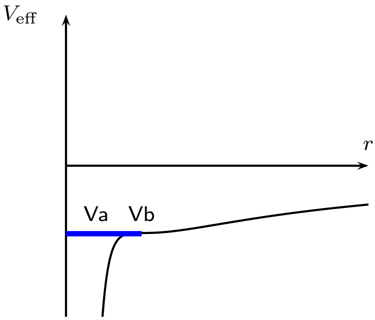
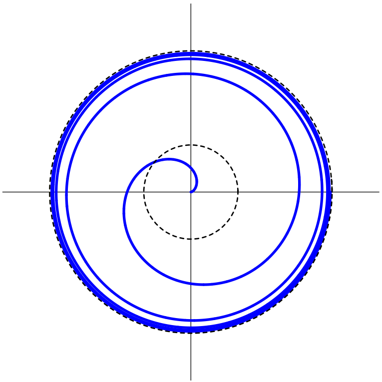
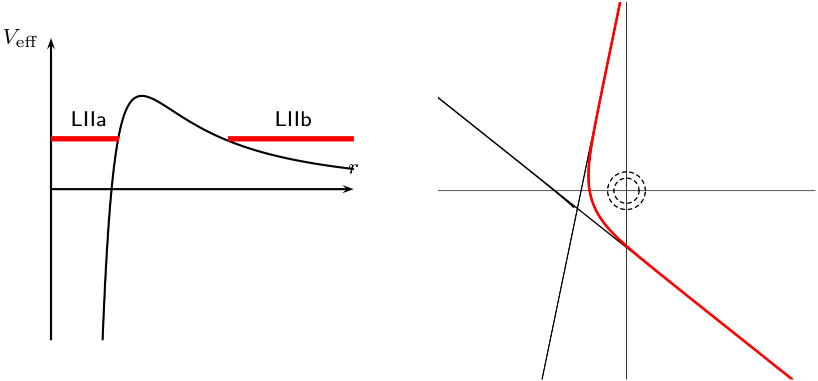




spiral — circle (ISCO)

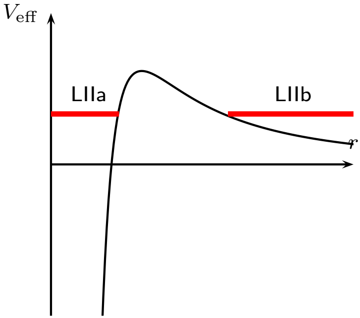
Light rays in Schwarzschild space–time

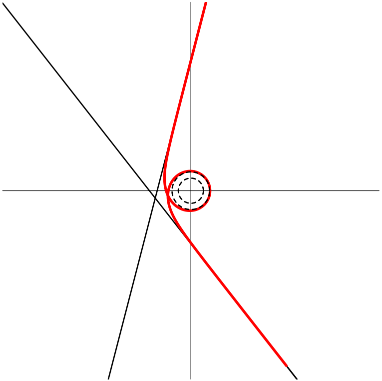


quasi–hyperbolic

deflection angle given by complete elliptic integral as function of $r_{\min}$

Light rays in Schwarzschild space–time





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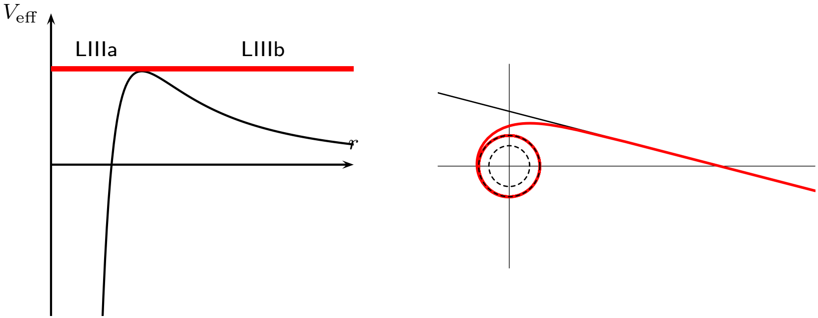
Light rays in Schwarzschild space–time



infinite quasi–hyperbolic spiral

Outline

Schwarzschild–de Sitter space–time

The Schwarzschild–de Sitter metric

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{1}{3}\Lambda r^2\right) dt^2 - \frac{1}{1 - \frac{2M}{r} + \frac{1}{3}\Lambda r^2} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2$$

Equations of motion

Conservation laws

$$E = g_{00} \frac{dt}{ds} \,, \qquad L = r^2 \frac{d\varphi}{ds}$$

With substitutions, e.g., $u = 2M/r$

$$d\frac{\varphi}{2} \quad = \quad \frac{du}{\sqrt{P_5(u)}}$$

and similar for t and s, with

$$g_2, g_3 \qquad \leftrightarrow \qquad L, E, M, \Lambda$$

Geodesic equation in Schwarzschild–de Sitter space–time

Effective equation of motion ($u = r_{\rm S}/r$)

$$\left(u\frac{du}{d\varphi}\right)^2 = u^5 - u^4 + \epsilon\lambda u^3 + \left(\lambda(\mu-\epsilon) + \frac{1}{3}\rho\right)u^2 + \frac{\epsilon}{3}\lambda\rho =: P_5(u)$$

with

$$\lambda = \left(\frac{r_{\rm S}}{L}\right)^2, \qquad \mu = E^2, \qquad \rho = \Lambda r_{\rm S}^2$$

- Polynomial of 5th order $\rightarrow$ beyond elliptic integral: hyperelliptic integral
- P_5 possesses at most 4 real positive zeros
- $\epsilon = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Λ has no influence on light propagation)
- $\Lambda = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$

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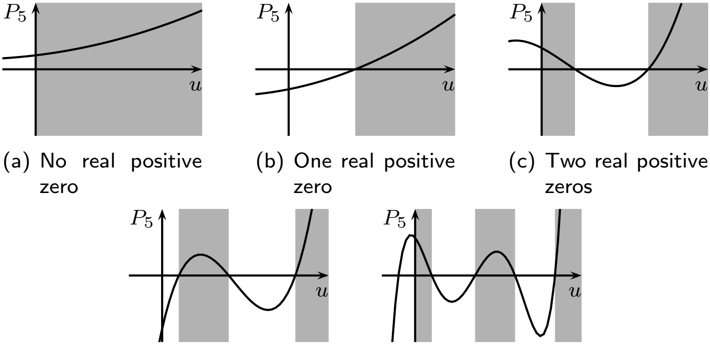
Orbits in Schwarzschild–de Sitter space–time



(a) No real positive zero

(b) One real positive zero

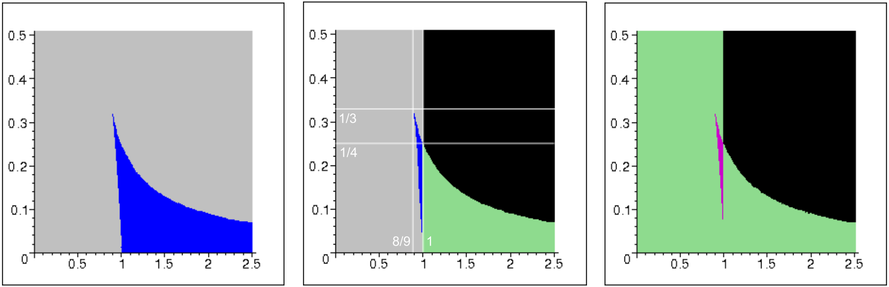
(c) Two real positive zeros

(d) Three real positive zeros

(e) Four real positive zeros

- Five possibilities of having real positive zeros of $P_5(u)$
- Zeros correspond to the positions for which $V_{\text{eff}} = E$
- Bound non–terminating, quasi–periodic orbits (planetary orbits) exist only for three or more positive zeros

Orbits in Schwarzschild–de Sitter space–time



(f) $\Lambda = -10^{-45}\text{km}^{-2}$ (g) $\Lambda = 0$ (h) $\Lambda = 10^{-45}\text{km}^{-2}$

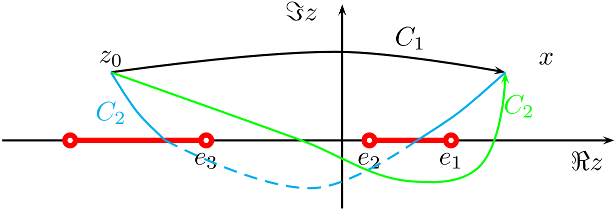
color	# zeros > 0	types of orbits
black	0	particle from infinity to singularity
gray	1	bound terminating orbits
green	2	bound terminating and escape orbits
blue	3	bound terminating and quasi–periodic bound orbits
violet	4	bound terminating, escape and quasi–periodic bound orbits

Analytic solution of geodesic equation in SdS–space–time

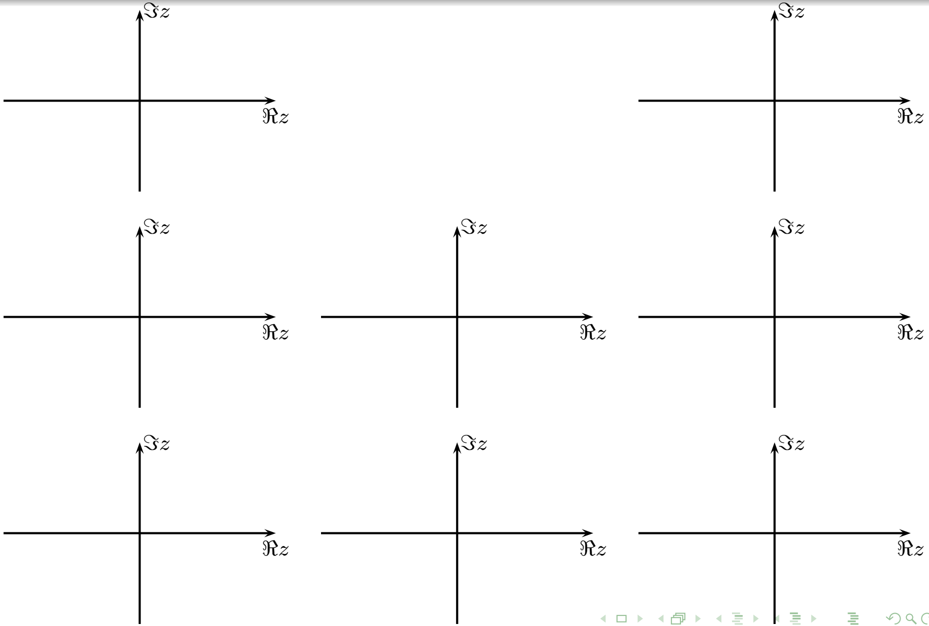
Separation of variables

$$\varphi - \varphi_0 = \int_{u_0}^{u} \frac{u' du'}{\sqrt{P_5(u')}}$$

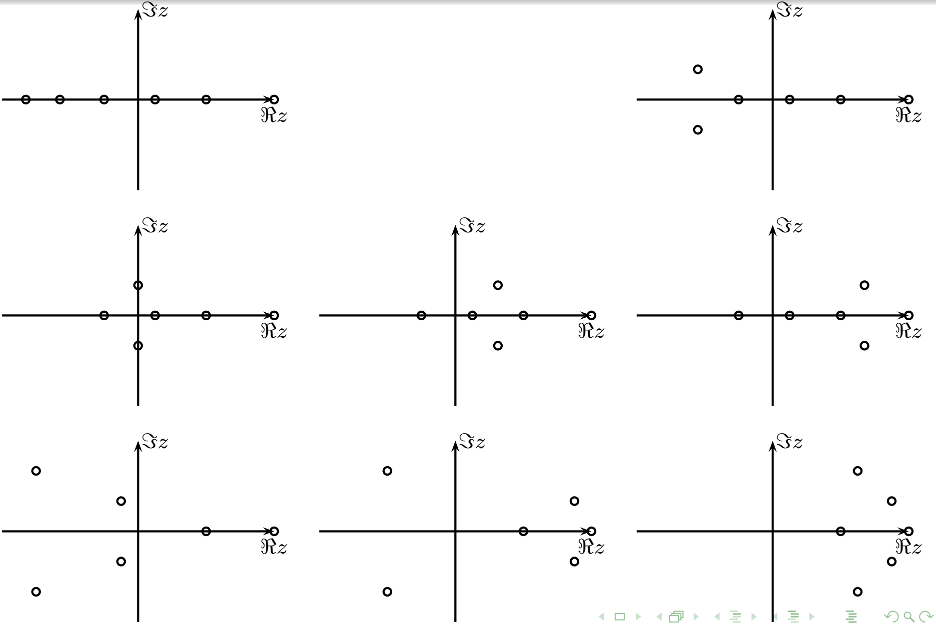
- Not well defined in complex plane
- Looked for: $u = u(\varphi) \leftrightarrow$ inversion problem
- Uniqueness of integration: u is function with 4 periods



Analytic solution of geodesic equation in SdS–space–time



Analytic solution of geodesic equation in SdS–space–time



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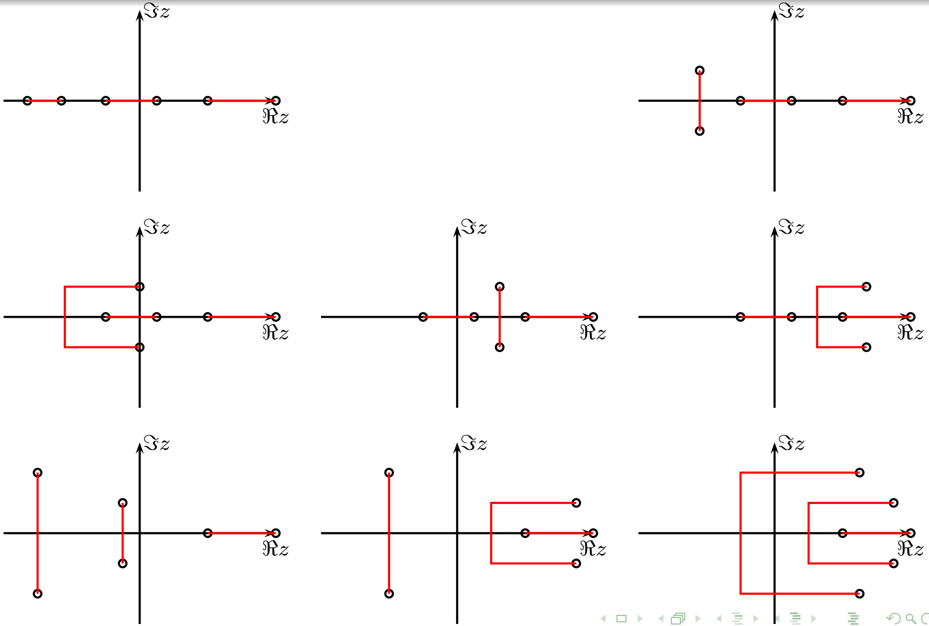
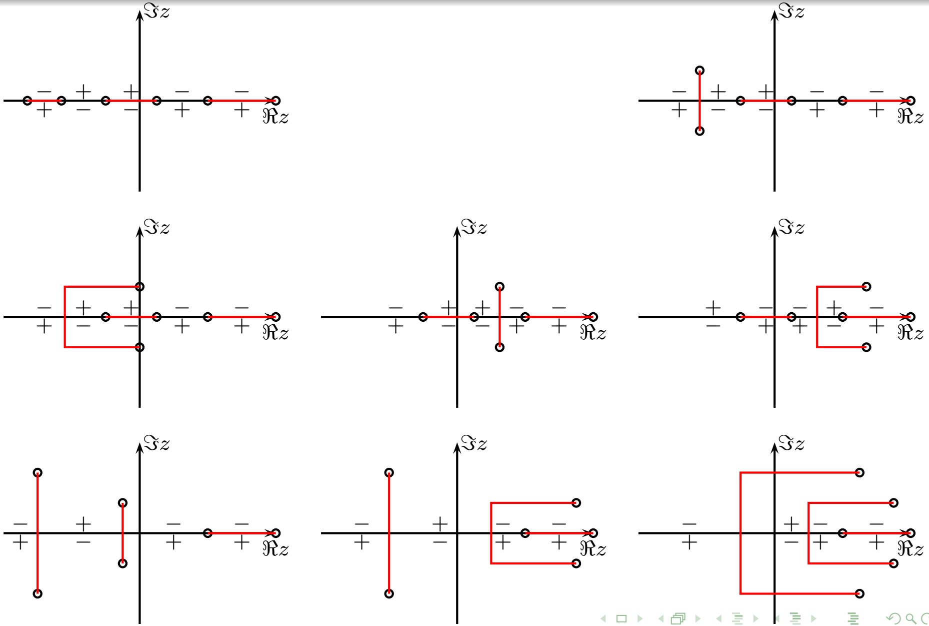


Analytic solution of geodesic equation in SdS–space–time



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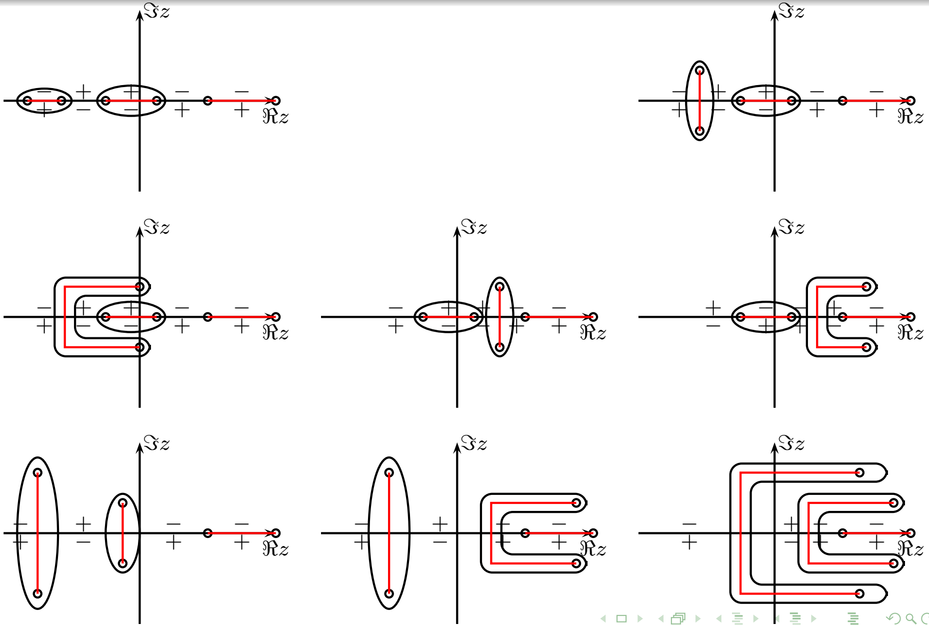

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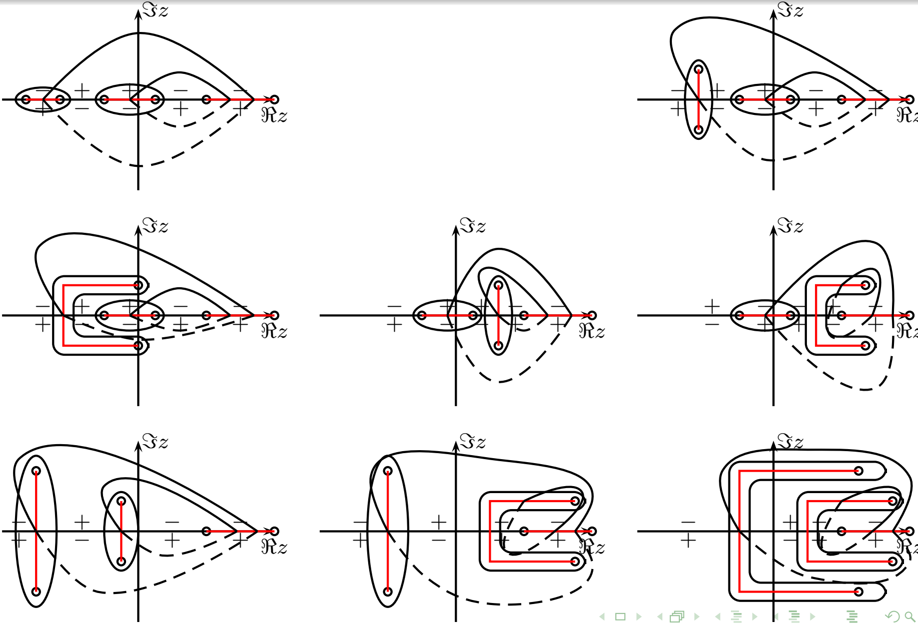

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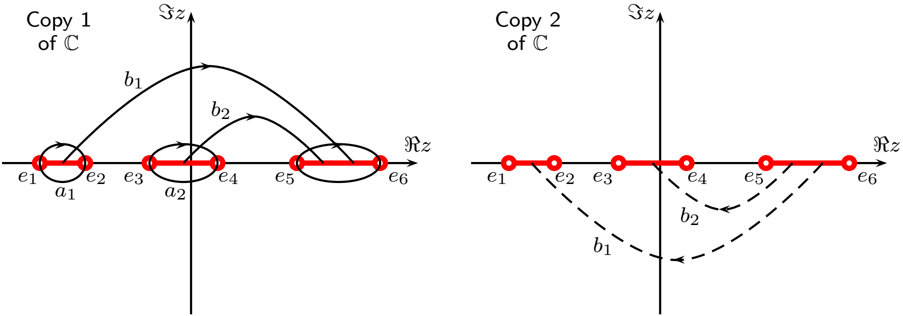

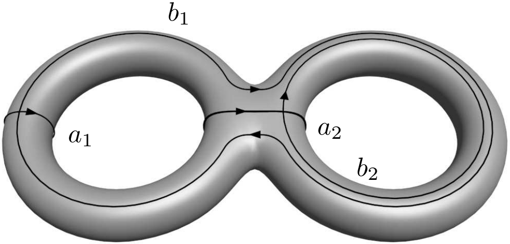

$P_5 \leftrightarrow X =$ pretzel

Analytic solution of geodesic equation in SdS–space–time

Holomorphic and associated meromorphic differentials

$$dz_1 := \frac{dx}{\sqrt{P_5(x)}}, \qquad dz_2 := \frac{x\,dx}{\sqrt{P_5(x)}}$$

$$dr_1 := \frac{3x^3 - 2x^2 + \lambda x}{4\sqrt{P_5(x)}} dx, \qquad dr_2 := \frac{x^2 dx}{4\sqrt{P_5(x)}}$$

Period matrices $(2\omega, 2\omega')$ and $(2\eta, 2\eta')$

$$2\omega_{ij} := \oint_{a_j} dz_i, \qquad 2\omega'_{ij} := \oint_{b_j} dz_i$$

$$2\eta_{ij} := -\oint_{a_j} dr_i, \qquad 2\eta'_{ij} := -\oint_{b_j} dr_i$$

Normalized differentials and their period matrix

$$d\vec{z} \to d\vec{v} = (2\omega)^{-1} d\vec{z}, \qquad (2\omega, 2\omega') \to (1_2, \tau) \quad \text{with} \quad \tau = \omega^{-1}\omega'$$

Analytic solution of geodesic equation in SdS–space–time

Preliminaries – definitions

- Theta function $\vartheta : \mathbb{C}^2 \to \mathbb{C}$ (for construction of functions with 4 periods)

$$\vartheta(\vec{z};\tau) := \sum_{\vec{m}\in\mathbb{Z}^2} e^{i\pi\vec{m}^t(\tau\vec{m}+2\vec{z})}$$

  - Periodicity: $\vartheta(\vec{z}+1_2\vec{n};\tau) = \vartheta(\vec{z};\tau)$
  - Quasi–periodicity: $\vartheta(\vec{z}+\tau\vec{n};\tau) = e^{-i\pi\vec{n}^t(\tau\vec{n}+2\vec{z})}\vartheta(\vec{z};\tau)$

- Theta function with characteristics $\vec{g},\vec{h} \in \frac{1}{2}\mathbb{Z}^2$

$$\vartheta[\vec{g},\vec{h}](\vec{z};\tau) := e^{i\pi\vec{g}^t(\tau\vec{g}+2\vec{z}+2\vec{h})}\vartheta(\vec{z}+\tau\vec{g}+\vec{h};\tau)$$

- Sigma function $\sigma(\vec{z}) = Ce^{-\frac{1}{2}\vec{z}^t\eta\omega^{-1}\vec{z}}\vartheta\left[\begin{pmatrix}1/2\\1/2\end{pmatrix},\begin{pmatrix}0\\1/2\end{pmatrix}\right]((2\omega)^{-1}\vec{z};\tau)$

- Generalized Weierstrass functions

$$\wp_{ij}(\vec{z}) = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\log\sigma(\vec{z}) = \frac{\partial_i\sigma(\vec{z})\partial_j\sigma(\vec{z}) - \sigma(\vec{z})\partial_i\partial_j\sigma(\vec{z})}{\sigma^2(\vec{z})}$$

Analytic solution of geodesic equation in SdS–space–time

Jacobi's inversion problem

Determine $\vec{u}$ for given $\vec{\varphi}$ from $\vec{\varphi} = \vec{A}_{u_0}(\vec{u})$ (Abel map), i.e.

$$\begin{aligned} \varphi_1 &= \int_{u_0}^{u_1} \frac{du}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{du}{\sqrt{P_5(u)}} \\ \varphi_2 &= \int_{u_0}^{u_1} \frac{udu}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{udu}{\sqrt{P_5(u)}} \end{aligned}$$

Solution of inversion problem

Solution $\vec{u}$ of inversion problem given by

$$\begin{aligned} u_1 + u_2 &= 4\wp_{22}(\vec{\varphi}) \\ u_1\, u_2 &= -4\wp_{12}(\vec{\varphi}) \end{aligned}$$

Our problem: two positions $\vec{u}$, two angles $\vec{\varphi}$ $\quad \longrightarrow$ requires reduction

Analytic solution of geodesic equation in SdS–space–time

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Our problem: two positions $\vec{u}$, two angles $\vec{\varphi}$ $\quad \longrightarrow$ requires reduction

Analytic solution of geodesic equation in SdS–space–time

Rewrite inversion problem in the form (based on Enolskii & Richter 2005)

$$\vec{\phi} = A_\infty(\vec{u}) \qquad \text{with} \qquad \vec{\phi} = \vec{\varphi} - \int_{u_0}^{\infty} d\vec{z}$$

Extraction of one component of $\vec{u}$ (namely u_2 in Jacobi inv. prob.) through a limit

$$u_1 = \lim_{u_2 \to \infty} \frac{u_1 u_2}{u_1 + u_2} = \frac{\sigma(\vec{\phi}_\infty)\partial_1\partial_2\sigma(\vec{\phi}_\infty) - \partial_1\sigma(\vec{\phi}_\infty)\partial_2\sigma(\vec{\phi}_\infty)}{(\partial_2\sigma)^2(\vec{\phi}_\infty) - \sigma(\vec{\phi}_\infty)\partial_2\partial_2\sigma(\vec{\phi}_\infty)}$$

with

$$\vec{\phi}_\infty = \lim_{u_2 \to \infty} \vec{\phi} = \vec{A}_\infty(\vec{u}_\infty)\,, \qquad \vec{u}_\infty = \begin{pmatrix} u_1 \\ \infty \end{pmatrix}$$

One u but still two φ: Lemma (Mumford 1983 – Riemann vanishing theorem): $(2\omega)^{-1}\vec{A}_\infty(\vec{u}_\infty) \in$ Theta–divisor $\Theta_{K_\infty} = \{\vec{z} \mid \vartheta(\vec{z} + \vec{K}_\infty) = 0\}$ with the vector of Riemann constants $\vec{K}_\infty = \tau \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$\Rightarrow \sigma(\vec{\phi}_\infty) = 0$, gives a relation between $\phi_{\infty,1}$ and $\phi_{\infty,2}$.

Analytic solution of geodesic equation in SdS–space–time

Define

$$\vec{\varphi}_\Theta := \begin{pmatrix} x \\ \varphi - \varphi_0' \end{pmatrix} \qquad \text{with} \qquad \varphi_0' = \varphi_0 + \int_{u_0}^{\infty} dz_2$$

and choose x so that $(2\omega)^{-1}\vec{\varphi}_\Theta \in \Theta_{K_\infty}$.

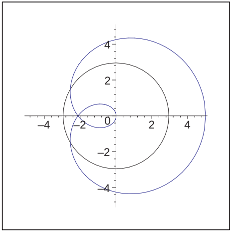
Solution (Hackmann & C.L. PRL 2008, PRD 2008)

Complete analytic solution of equation of motion in Schwarzschild–de Sitter space–time

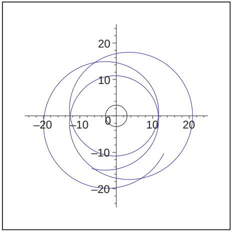
$$r(\varphi) = \frac{r_S}{u(\varphi)} = -r_S \frac{\partial_2 \sigma(\vec{\varphi}_\Theta)}{\partial_1 \sigma(\vec{\varphi}_\Theta)} \qquad \sigma(\vec{\varphi}_\Theta) = 0$$

New result

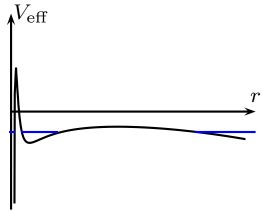
Schwarzschild–de Sitter: Orbits



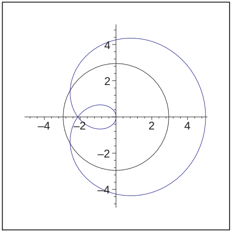
(i) $\Lambda = 0$, $r_0 = 5.010$km



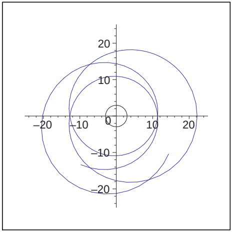
(j) $\Lambda = 0$, $r_0 = 20.951$km



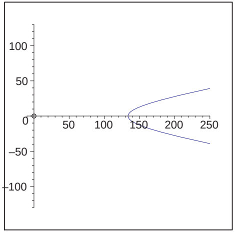
(k) effective potential



(l) $\Lambda = 10^{-5}\text{km}^{-2}$, $r_0 = 5.013$km

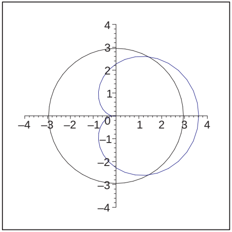


(m) $\Lambda = 10^{-5}\,\text{km}^{-2}$, $r_0 = 22.185$km

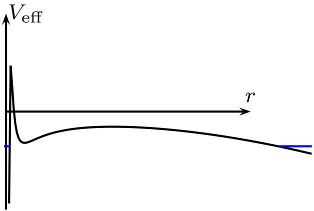


(n) $\Lambda = 10^{-5}\text{km}^{-2}$, $r_0 = 133.60$km

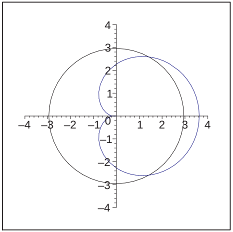
Schwarzschild–de Sitter: Orbits



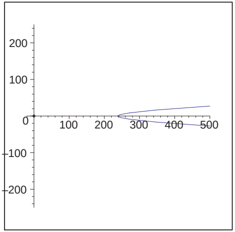
(o) $\Lambda = 0$, $r_0 = 3.625$km



(p) effective potential

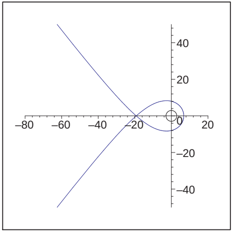


(q) $\Lambda = 10^{-5}$km^{-2}, $r_0 = 3.625$km

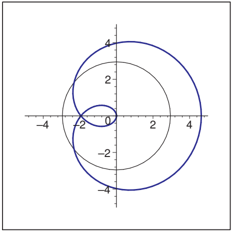


(r) $\Lambda = 10^{-5}$km^{-2}, $r_0 = 237.61$km

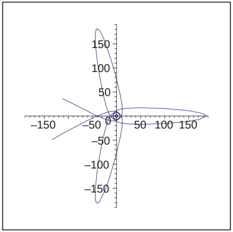
Schwarzschild–de Sitter: Orbits



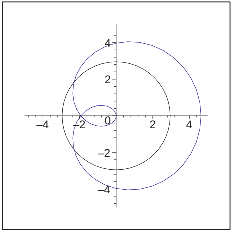
(s) $\Lambda = 0$, $r_0 = 6.776\text{km}$



(t) $\Lambda = 0$, $r_0 = 4.642\text{km}$



(u) $\Lambda = -10^{-5}\text{km}^{-2}$, $r_0 = 185.37\text{km}$



(v) $\Lambda = -10^{-5}\text{km}^{-2}$, $r_0 = 4.639\text{km}$

Application to Pioneer

Influence of Λ on Pioneer satellites (orbital parameters from Nieto & Anderson 2005)

Pioneer 10: $\mu = 1.000\,000\,001\,43\,,\quad \lambda = 2.855\,572\,373\,82 \cdot 10^{-9}$
Pioneer 11: $\mu = 1.000\,000\,001\,22\,,\quad \lambda = 1.340\,740\,574\,59 \cdot 10^{-9}$

- For given angle φ: $r_\Lambda(\varphi) - r_{\Lambda=0}(\varphi) \approx 10^{-3}$ m
- For given distance $r = 65$ AU: $r\varphi_\Lambda(r) - r\varphi_{\Lambda=0}(r) \approx 10^{-5}$ m

$\Rightarrow$ Form of the Pioneer orbits practically does not change.

Cosmological constant cannot be origin of Pioneer anomaly.
In principle we need also $\varphi = \varphi(t)$ and Doppler tracking.

Post–Schwarzschild of perihelion shift

Perihelion shift in Schwarzschild–de Sitter (for bound orbit, Kraniotis & Whitehouse, CQG 2003)

$$\delta\varphi_{\text{perihelion}} = 2\pi - \omega_{22} = 2\pi - \oint \frac{x dx}{\sqrt{P_5(x)}}$$

Approximation to first order in Λ

$$\frac{x}{\sqrt{p_5(x)}} = \frac{1}{\sqrt{P_3(x)}} - \frac{2}{3} m^2 \frac{x^2 + \lambda}{x^2 P_3(x) \sqrt{P_3(x)}} \Lambda + \mathcal{O}(\Lambda^2)$$

with $P_3(x) = x^3 - x^2 + \lambda x + \lambda(\mu - 1)$ Schwarzschild polynomial
This has to be integrated: gymnastics in elliptic integration

Post–Schwarzschild of perihelion shift

result

$$\oint_{a_2} \frac{x dx}{\sqrt{P_5(x)}} = \oint_{a_2} \frac{1}{\sqrt{P_3(x)}} - \frac{2}{3}\Lambda m^2 \oint_{a_2} \frac{x^2+\lambda}{x^2 P_3(x)\sqrt{P_3(x)}} dx + \mathcal{O}(\Lambda^2)$$

$$= \omega_1 + \Lambda \frac{m^2}{96}\left(\sum_{j=1}^{3} \frac{\eta_1 + \omega_1 z_j}{(\wp''(\rho_j))^2}\left(1 + \frac{\lambda}{\left(4z_j + \frac{1}{3}\right)^2}\right)\right.$$

$$\left. + \lambda\left(\frac{2\eta_1 - \frac{1}{6}\omega_1}{16(\wp'(u_0))^2} + \frac{6}{16}\frac{\wp''(u_0)}{(\wp'(u_0))^5}\left(\zeta(u_0) - \eta_1 u_0\right)\right)\right)$$

$$+\mathcal{O}(\Lambda^2)$$

- Needs introduction of $r_{\min}$ and $r_{\max}$ or a and e for interpretation
- Needs relativistic approximation for interpretation

Outline

Further space–times: Higher dimensions

Formalism can be further applied to some Reissner–Nordström–(anti-)de Sitter space–times in higher dimensions

$$ds^2 = \alpha dt^2 - \alpha^{-1} dr^2 - r^2 d\Omega^2_{d-2}$$

with

$$\alpha = 1 - \frac{m^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2$$

Equation of motion

$$\begin{aligned}\left(\frac{dr}{d\varphi}\right)^2 &= \frac{r^4}{L^2}\frac{1}{g_{rr}g_{tt}}\left(E^2 - g_{tt}\left(\epsilon + \frac{L^2}{r^2}\right)\right) \\ &= \frac{r^4}{L^2}\left(E^2 - \left(1 - \frac{m^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right)\right)\end{aligned}$$

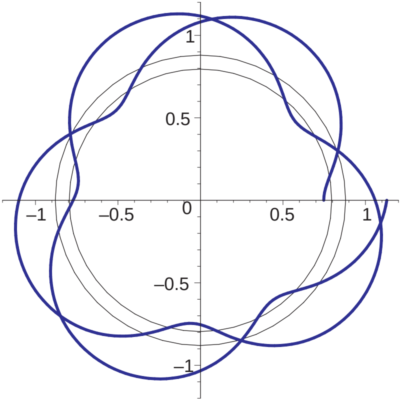
Substitution $u = \dfrac{m}{r}$, systematic discussion and study of all subcases $\longrightarrow$ ...

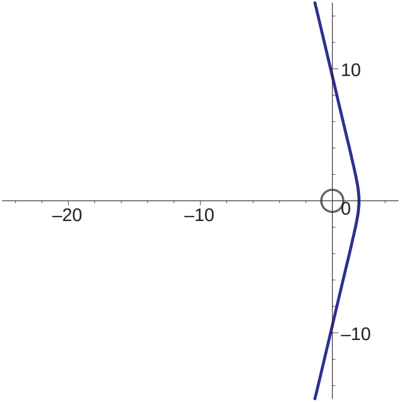
Analytically solvable cases

Space–time \ Dimension	4	5	6	7	8	9	10	11	$d \geq 12$
Schwarzschild	$+$	$+$	$\oplus$	$+$	$-$	$\oplus$	$-$	$\oplus$	$-$
Schwarzschild–de Sitter	$\oplus$	$+$	$-$	$+$	$-$	$\oplus$	$-$	$\oplus$	$-$
Reissner–Nordström	$+$	$+$	$-$	$\oplus$	$-$	$-$	$-$	$-$	$-$
Reissner–Nordström–de Sitter	$\oplus$	$+$	$-$	$\oplus$	$-$	$-$	$-$	$-$	$-$

(Hackmann, Kagramanova, Kunz & CL, PRD 2008)

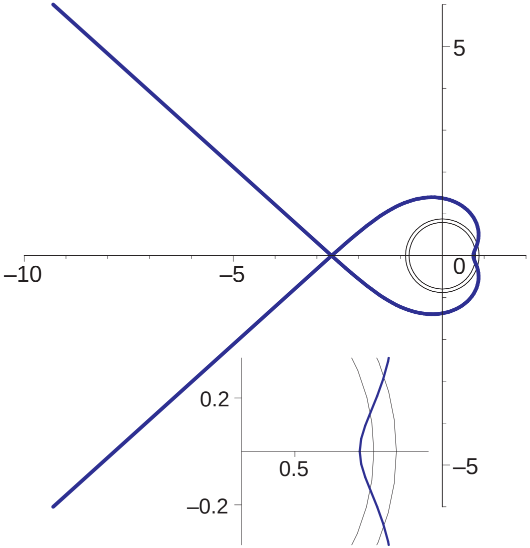
Reissner–Nordström in 7 dimensions





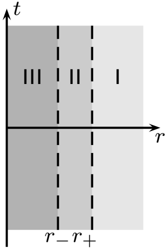
Traversing different universes: measurement of Perihelion shift?

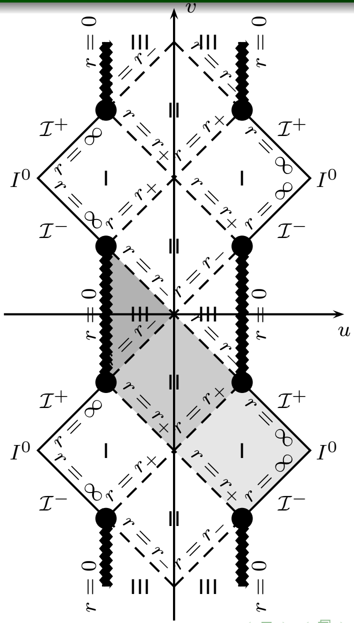
Reissner–Nordström in 7 dimensions



Escape orbit: escape into a different universe

Reissner-Nordström space–time





Outline

Kerr space–time

Kerr–de Sitter metric

rotating axially symmetric black hole in space–time with cosmological constant

$$ds^2 = \frac{\Delta_r - \Delta_\theta a^2 \sin^2\theta}{p^2} dt^2 + \sin^2\theta \frac{\Delta_r a^2 \sin^2\theta - (r^2+a^2)^2}{p^2} d\phi^2 + 2a\sin^2\theta \frac{r^2+a^2-\Delta_r}{p^2} dt d\phi - \frac{p^2}{\Delta_r} dr^2 - p^2 d\theta^2$$

$$\Delta_r = r^2 + a^2 - 2mr \qquad p^2 = r^2 + a^2\cos^2\theta$$

- ϑ–motion elentary integral, period $\omega^\vartheta \leftrightarrow \vartheta_{\min}, \vartheta_{\max}$
- r–motion Weierstraß elliptic integral, periods ω_1^r, $\omega_2^r \leftrightarrow r_{\min}, r_{\max}$

Chandrasekhar 1983; O'Neill 1995; Fujita & Hikida, CQG 2009

Outline

NUT–de Sitter space–time

NUT–de Sitter metric

$$ds^2 = \frac{\Delta_r}{\rho^2}\left(dt - 2l\cos\vartheta d\phi\right)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \rho^2\left(d\vartheta^2 + \sin^2\theta d\phi^2\right)$$

$\rho^2 = r^2 + l^2$, $\Delta_r = -\frac{\Lambda}{3}r^4 + r^2(1-2l^2\Lambda) - 2Mr + l^2\left(\Lambda l^2 - 1\right)$
$M =$ mass, $l =$ NUT charge (gravitomagnetic mass), $\Lambda =$ cosmological constant

Equations of motion

$$\frac{dr}{d\tau} = \sqrt{R_6} \quad \text{with} \quad R_6 = (r^2+l^2)^2E^2 - \Delta_r(m^2r^2 + L^2 + K)$$

$$\frac{d\vartheta}{d\tau} = \sqrt{\Theta} \quad \text{with} \quad \Theta = K - m^2l^2 - \frac{1}{\sin^2\vartheta}\left((4E^2l^2 + L^2)\cos^2\vartheta - 4ElL\cos\vartheta\right)$$

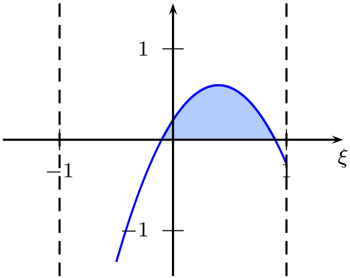
$$\frac{d\phi}{d\vartheta} = \frac{1}{\sqrt{\Theta}}\frac{L - 2lE\cos\vartheta}{\sin^2\vartheta}$$

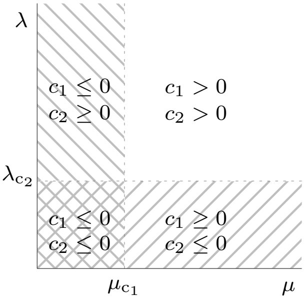
$$\frac{dt}{d\tau} = E\frac{\rho^4}{\Delta_r} + 2l\cos\vartheta\frac{L - 2lE\cos\vartheta}{\sin^2\vartheta} \qquad \tau = \text{ Mino time}$$

NUT–de Sitter: ϑ–motion

with $\xi := \cos\vartheta$

$$\left(\frac{d\xi}{d\tau}\right)^2 = \lambda^2 \left(A - B^2 - (A+1)\xi^2 + 2B\xi\right) = P_2(\xi)$$





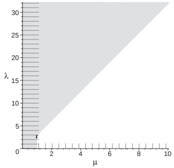
$B = 0$: P_2 symmetric
$A > 0$: P_2 possesses zeros

restrictions on λ and μ for having $P_2(\xi) \geq 0$

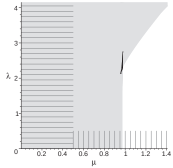
can be solved by elementary function

NUT–de Sitter: r–motion

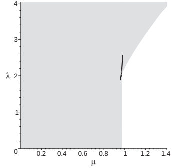
Effective potential for r-motion given by $R_6 \rightarrow \lambda - \mu$–diagrams



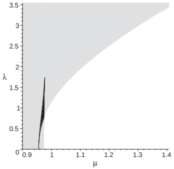
(w) $\kappa = -1.0$



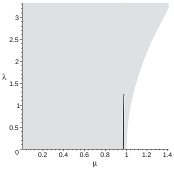
(x) $\kappa = 0.0$



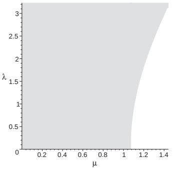
(y) $\kappa = 1.0 < \kappa_{\text{crit}_1}$



(z) $\kappa = \kappa_{\text{crit}_1}$



() $\kappa_{\text{crit}_1} < \kappa < \kappa_{\text{crit}_2}$



() $\kappa_{\text{crit}_2} < \kappa$

Figure: λ–μ diagrams for $n = 0.5$, $\tilde{\Lambda} = 8.7 \cdot 10^{-5}$. Dashed regions are forbidden from ϑ–equation. Black: 4 zeros. Gray: 2 zeros, White: No zero.

NUT–de Sitter: r–motion

analytical solution

$$r(\tau) = f\left(-\frac{\sigma_1(\vec{\tau}_\infty)}{\sigma_2(\vec{\tau}_\infty)}\right) \quad \text{with} \quad \vec{\vec{\tau}}_{\infty,z_2} = \begin{pmatrix} \int_{x_{\rm in}}^{x_1} dz_1 - \int_{x_{\rm in}}^{\infty} dz_1 \\ \tau - \tau' \end{pmatrix} \quad ,$$

where $\tau' = \tau_{\rm in} + \int_{x_{\rm in}}^{\infty} dz_2$

NUT–de Sitter: φ–motion

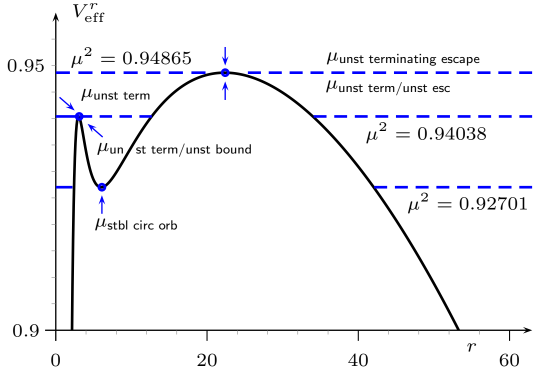
Solution to the φ–equation:

$$\varphi = \frac{1}{2}\left(\frac{\lambda+2\mu n}{|\lambda+2\mu n|}\arctan\frac{1-uB_1}{\sqrt{1-u^2}\sqrt{B_1^2-1}} - \frac{\lambda-2\mu n}{|\lambda-2\mu n|}\arctan\frac{1-uB}{\sqrt{1-u^2}\sqrt{B^2-1}}\right)$$

with $u = \alpha + \beta x$

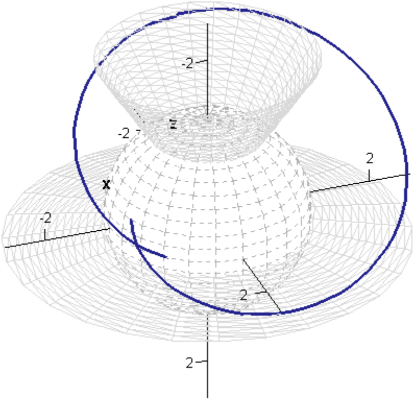
NUT–de Sitter: effective Potential

$n = 0.5$, $\mathcal{K} > 0$

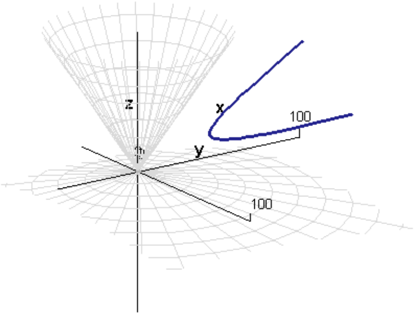


Remark:
$\mathcal{K} > 0$: $\theta \subset (0, \pi)$
$\mathcal{K} = 0$: $\theta \in [0, \frac{\pi}{2}]$.

Orbits in NUT–de Sitter space–time

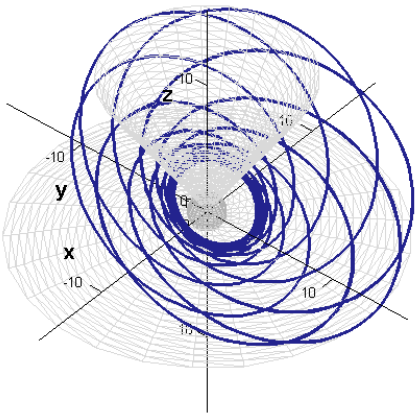


terminating orbit

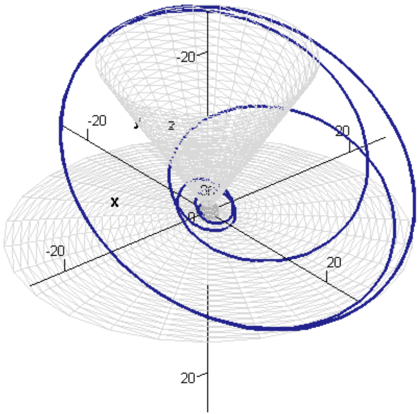


escape orbit

Orbits in NUT–de Sitter space–time

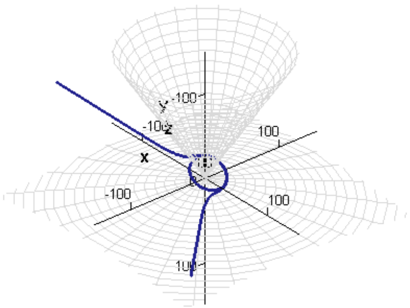


bound orbit

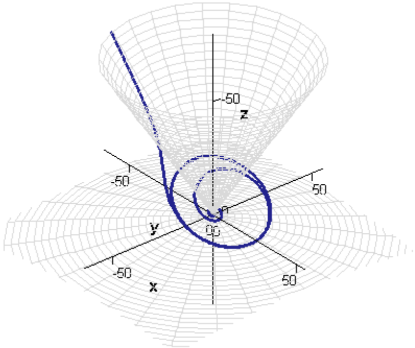


bound orbit

Orbits in NUT–de Sitter space–time

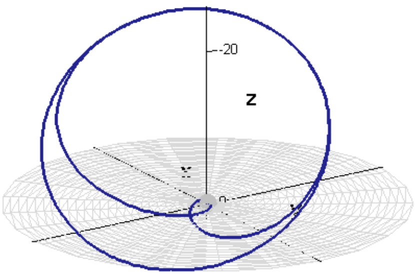


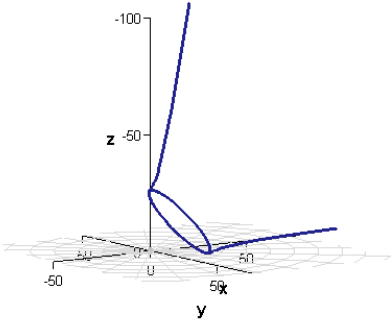
quasi–hyperbolic orbit



terminating inspiral orbit

Orbits in NUT–de Sitter space–time



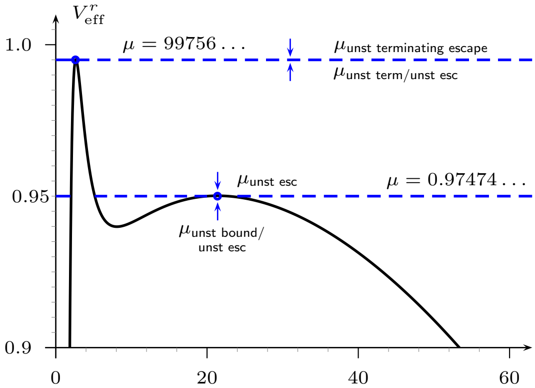


terminating orbit (for $\theta \in [0, \frac{\pi}{2}]$)

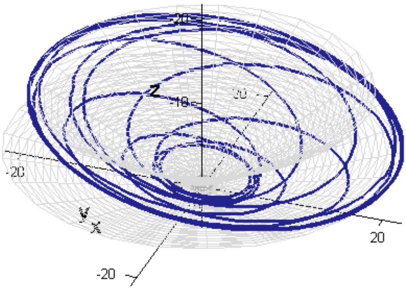
quasihyperbolic escape orbit (for $\theta \in [0, \frac{\pi}{2}]$)

NUT–de Sitter: effective Potential

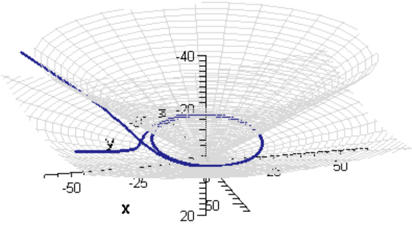
$n = 0.5$, $\mathcal{K} < 0$



Orbits in NUT–de Sitter space–time

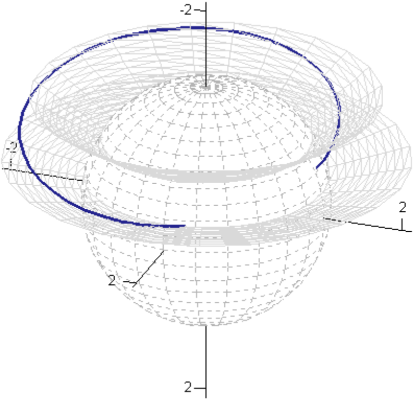


bound orbit ($\theta \in (0, \frac{\pi}{2})$)

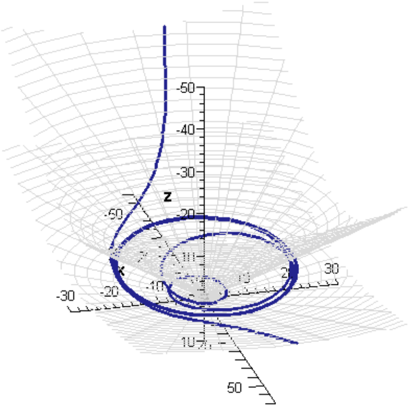


quasihyperbolic escape orbit ($\theta \in (0, \frac{\pi}{2})$)

Orbits in NUT–de Sitter space–time

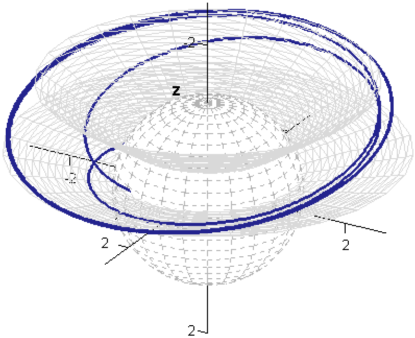


terminating orbit ($\theta \in (0, \frac{\pi}{2})$)

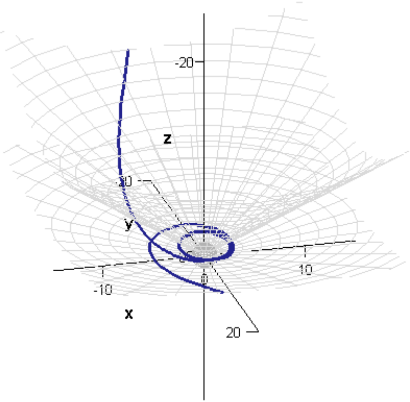


quasihyerbolic escape orbit
($\theta \in (0, \frac{\pi}{2})$)

Orbits in NUT–de Sitter space–time

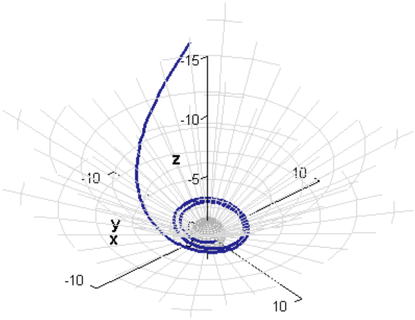


terminating orbit ($\theta \in (0, \frac{\pi}{2})$)



quasihyerbolic escape orbit ($\theta \in (0, \frac{\pi}{2})$)

Orbits in NUT–de Sitter space–time



(Hackmann, Kagramanova, Kunz & C.L. 2009)

terminating inspiral orbit ($\theta \in (0, \frac{\pi}{2})$)

Outline

Kerr–de Sitter space–time

Kerr–de Sitter metric

rotating axially symmetric black hole in space–time with cosmological constant

$$ds^2 = \frac{\Delta_r - \Delta_\vartheta a^2 \sin^2\vartheta}{\chi^2 p^2} dt^2 + \sin^2\vartheta \frac{\Delta_r a^2 \sin^2\theta - \Delta_\vartheta (r^2 + a^2)^2}{\chi^2 p^2} d\varphi^2$$
$$+ 2a \sin^2\vartheta \frac{\Delta_\vartheta (r^2 + a^2) - \Delta_r}{\chi^2 p^2} dt d\phi - \frac{p^2}{\Delta_r} dr^2 - \frac{p^2}{\Delta_\vartheta} d\vartheta^2$$

$$\Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right)(r^2 + a^2) - 2mr \qquad \Delta_\vartheta = 1 + \frac{a^2\Lambda}{3}\cos^2\vartheta$$

$$\chi = 1 + \frac{a^2\Lambda}{3} \qquad p^2 = r^2 + a^2\cos^2\vartheta$$

Kerr–de Sitter space–time

Equations of motion

$$\left(\frac{dx}{d\tau}\right)^2 = E^2 r_S^2 (\chi^2 (x^2 + \alpha^2 - \alpha\mathcal{D})^2 - \Delta_x (\delta_2 x^2 + \kappa)) =: R_6$$

$$\left(\frac{d\vartheta}{d\tau}\right)^2 = E^2 r_S^2 \left(\Delta_\vartheta (\kappa - \delta_2 \alpha^2 \cos^2\vartheta) - \frac{\chi^2}{\sin^2\vartheta} \left(\alpha \sin^2\vartheta - \mathcal{D}\right)^2\right)$$

$$\frac{d\varphi}{d\tau} = \chi^2 E r_S \left[\frac{\alpha}{\Delta_x}\left((x^2 + \alpha^2) - \alpha\mathcal{D}\right)\right) - \frac{1}{\Delta_\vartheta \sin^2\vartheta} \left(\alpha \sin^2\vartheta - \mathcal{D}\right)\right]$$

$$\frac{dt}{d\tau} = \chi^2 E r_S^2 \left[\frac{x^2 + \alpha^2}{\Delta_x}\left((r_n^2 + \alpha^2) - \alpha\mathcal{D}\right) - \frac{\alpha}{\Delta_\theta} \left(\alpha \sin^2\theta - \mathcal{D}\right)\right]$$

- τ = 'Mino time' coupled to the proper time $\tau = \int_0^s ds\, p^2$
- $x := \frac{r}{r_S}$, $\alpha := \frac{a}{r_S}$, $\mathcal{D} := \frac{E}{L r_S}$, $\lambda := \frac{1}{3}\Lambda r_S^2$, $\kappa := \frac{K}{E^2 r_S^2}$, $\delta_2 = \frac{1}{E^2}$.

Kerr–de Sitter space–time: ϑ–motion

With $\xi = \cos\vartheta$: ξ–motion based on P_4
$\rightarrow$ Weierstraß elliptic function

$$\vartheta(s) = \arccos\left(\sqrt{\frac{a_3}{4\wp(2Er_S s - \tilde{s}_0; g_2, g_3) - \frac{a_2}{3}}}\right)$$

where $\tilde{s}_0 = \tilde{s}_0(s_0, \theta_0)$ and a_2, a_3, g_2 and g_3 some constants.

Kerr–de Sitter space–time: r–motion

based on $P_6 \rightarrow$ hyperelliptic function

$$r(s) = -r_S \frac{\sigma_2}{\sigma_1}\left(\begin{pmatrix} f(Er_S\sqrt{a_5}s - \tilde{s}_0) \\ Er_S\sqrt{a_5}s - \tilde{s}_0 \end{pmatrix}\right) + l\,.$$

where f such that $\sigma((f(z), z)^t) = 0$, $\tilde{s}_0 = \tilde{s}_0(s_0, r_0)$, l a zero of R_6 and a_5 a constant.

Kerr–de Sitter space–time: φ–motion

Solution for $\varphi(s)$ consists of two parts

$$\varphi(s) = \chi^2(\varphi_r(s) - \varphi_\vartheta(s)) + \phi_0$$

requires hyperelliptic integrals of third kind

$$\varphi_\theta(s) = c_0(s - s_0) - \sum_{i=1}^{2} c_i \left[2Er_S\zeta(v_i)(s - s_0) + \log\frac{\sigma(v - v_i)}{\sigma(v_0 - v_i)} \right]$$

where $v = 2Er_S s - \tilde{s}_0$, v_i and c_i some constants

$$\begin{aligned}\varphi_r(s) &= C_1 Er_S(s - s_0) + C_0(f(w) + f(-Er_S\sqrt{a_5}s_0 + \tilde{s}_0)) \\ &\quad + \sum_{i=1}^{4} C_{2,i} \left(\log\frac{\sigma((f(w), w)^t - \vec{k}_{4,i})}{\sigma((f(w), w)^t - \vec{k}_{5,i})} - k_{1,i} - (k_{2,i}, k_{3,i}) \begin{pmatrix} f(w) \\ w \end{pmatrix} \right)\end{aligned}$$

where $w = Er_S\sqrt{a_5}s - \tilde{s}_0$, , $k_{j,i}$ are constant

(Hackmann, Kagramanova, Kunz & C.L. 2009)

Kerr–de Sitter space–time: Orbits

Bound orbit for $r_S = 1$, $\alpha = 0.1$, $\Lambda = 10^{-5}$, $E = 0.98$, $L = 1$, and $K = 6$.

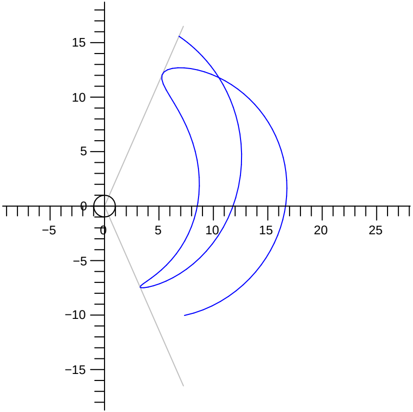


Figure: Plot in the (r, θ) plane. Grey lines denote extremal θ values.

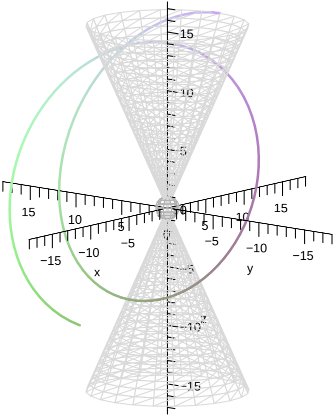


Figure: 3d plot. Grey cones denote extremal θ values and grey sphere the horizon.

Outline

General statements

Theorem: Separability

The Hamilton–Jacobi equation for the geodesic equation is separable if and only if space–time is of Petrov type D without acceleration (Demianski & Francaviglia, JPA 1981)

Theorem: Petrov type D

The general type D Petrov space–times are exhausted by the Plebański–Demiański solutions

Theorem: Integration

The geodesic equation in a general Plebański–Demiański space–time without acceleration can be integrated using the method of hyperelliptic integrals

All what is separable can be explicitely solved analytically!

Outline

Summary

Analytic solution / integration of all separable cases

New solution for

- Schwarzschild–de Sitter
- Kerr–de Sitter
- NUT–de Sitter
- general Plebański–Demiański

Solutions given by Kleinian sigma functions restricted to the Theta–divisor

Outlook

- Thorough discussion of observables
- Motion in gravitational multipole fields
- Comparison with LAGEOS, LARES and other geodesy missions
- Application to EMRIs – generation of gravitational waves
- Application to effective one–body formalism

- Orbits for particles with spin (C.L. & Schaffer 2009)
- Orbits in Finsler space-time (C.L. & Perlick 2009)

Last remark on LARES

Lares = Roman God for protection

LARES = Protection for

- Fundamental Physics in Space
  lares fisica fundamentale spatiale
- all geodesy missions
  lares geodesica
- good science in general
  lares sciencia
- ...

Thanks

Thank you !

Thanks to

- German Research Foundation DFG
- DFG Center of Excellence QUEST
- German Aerospace Center DLR