Tidal Tensor approach to Gravitomagnetism

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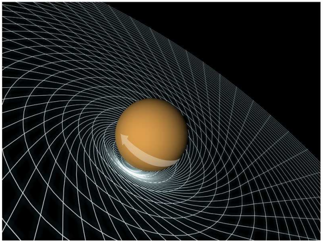
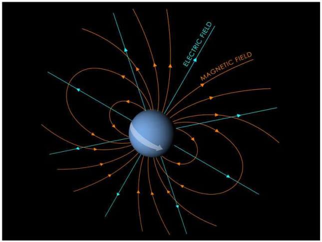




The Approach Based on Tidal Tensors — Motivation

Propose a new approach based on general, exact, and covariant equations, with which we aim to:

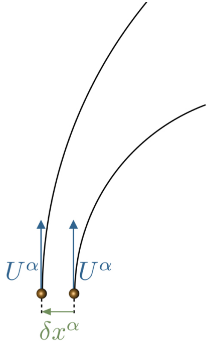
- Clarify the relationship between the various gravito-electromagnetic analogies found in literature, as well as issues within each of them;
- Search for the underlying principle behind the physical analogy between linearized Gravity and Electromagnetism;
- Introduce a new formalism allowing for a transparent comparison between the two interactions, and to study the physical similarities between them at a more fundamental level.

The Approach Based on Tidal Tensors — Guiding Principle

- A transparent comparison between the electromagnetic and gravitational interactions must be based on quantities common to both theories;
- the electromagnetic interaction is based on forces;
- but gravity is pure geometry, the only physical, covariant forces are tidal forces (tidal forces are the true signature of gravity!)
- Therefore: tidal forces should be the starting point for our approach.

Electric-type Tidal Tensors

Tidal forces are described in an invariant way through the wordline deviation equations:



- Electromagnetic

$$\frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha{}_\gamma \delta x^\gamma \,, \qquad E^\alpha{}_\gamma \equiv F^\alpha{}_{\beta;\gamma} U^\beta$$

- Gravitational (geodesic deviation)

$$\frac{D^2 \delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha{}_\gamma \delta x^\gamma \,, \qquad \mathbb{E}^\alpha{}_\gamma \equiv R^\alpha{}_{\beta\gamma\sigma} U^\beta U^\sigma$$

which yield the acceleration of the vector δx^α connecting two particles with the *same* (Ciufolini, 1986) 4-velocity U^α — and the same q/m ratio in the electromagnetic case.
(Notation: $F_{\alpha\beta} \equiv$ Maxwell tensor, $R_{\alpha\beta\gamma\sigma} \equiv$ Riemann tensor)

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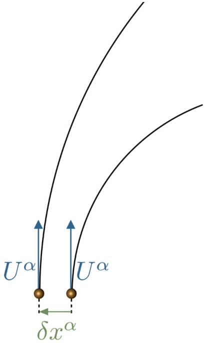

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- Suggests the physical analogy: $E_{\alpha\beta} \longleftrightarrow \mathbb{E}_{\alpha\beta}$
- $E_{\alpha\beta}$ is the covariant derivative of the electric field $E^\alpha = F^{\alpha\mu} U_\mu$ measured by the observer with (fixed) 4-velocity U^α;

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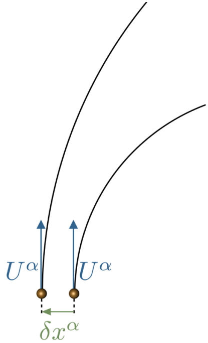

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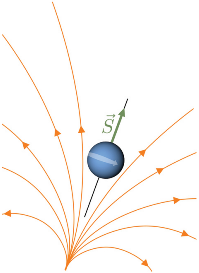
- Suggests the physical analogy: $E_{\alpha\beta} \longleftrightarrow \mathbb{E}_{\alpha\beta}$
- Hence:
  - $E_{\alpha\beta} \equiv$ *electric tidal tensor*; $\mathbb{E}_{\alpha\beta} \equiv$ *gravito-electric tidal tensor*.

Magnetic-type Tidal Tensors

The electromagnetic force exerted on a magnetic dipole and the gravitational force causing the non-geodesic motion of a spinning test particle are analogous tidal effects:



- Electromagnetic Force on a Magnetic Dipole (Covariant form for $\vec{F}_{EM} = \nabla(\vec{\mu}.\vec{B})$)

$$F^{\alpha}_{EM} \equiv \frac{DP^{\alpha}}{D\tau} = \sigma B_{\gamma}{}^{\alpha} S^{\gamma}\,, \qquad B^{\alpha}{}_{\gamma} \equiv \star F^{\alpha}{}_{\beta;\gamma} U^{\beta}$$

- Gravitational Force on a Gyroscope (Papapetrou-Pirani equation)

$$F^{\alpha}_{G} \equiv \frac{DP^{\alpha}}{D\tau} = -\mathbb{H}_{\gamma}{}^{\alpha} S^{\gamma}\,, \quad \mathbb{H}^{\alpha}{}_{\gamma} \equiv \star R^{\alpha}{}_{\beta\gamma\sigma} U^{\beta} U^{\sigma}$$

Notation: $\star \equiv$ Hodge dual; $S^{\alpha} \equiv$ spin 4-vector.

- Suggests the physical analogy: $B_{\alpha\beta} \longleftrightarrow \mathbb{H}_{\alpha\beta}$

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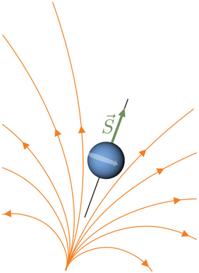

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- $B_{\alpha\beta}$ measures the tidal effects produced by the magnetic field $B^{\alpha} = \star F^{\alpha\mu} U_{\mu}$ seen by the observer of 4-velocity U^{α}.

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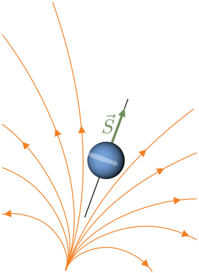

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- Hence: $B_{\alpha\beta} \equiv$ *magnetic tidal tensor;* $\mathbb{H}_{\alpha\beta} \equiv$ *gravito-magnetic tidal tensor.*

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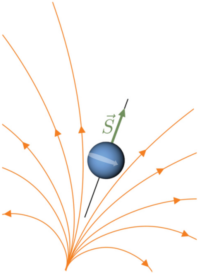

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- $\sigma = \mu/S \equiv$ gyromagnetic ratio $\Rightarrow$ equals 1 for gravity $\Rightarrow \vec{\mu} \leftrightarrow \vec{S}$

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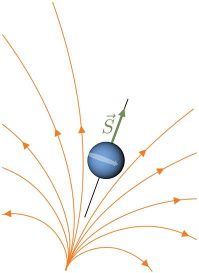

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- Relative minus sign: mass/charges of the same sign attract/repel one another $\Rightarrow$ antiparallel charge/mass currents repel/attract.

Maxwell's Equations: Tidal tensors and sources

- Maxwell's equations may be cast, in an explicitly covariant form, as equations for tidal tensors and sources:

Maxwell's Equations: Tidal tensors and sources

Maxwell Equations	
Tidal tensor form	Non-covariant form
$E^{\alpha}{}_{\alpha} = 4\pi\rho_c$	$\nabla.\vec{E} = 4\pi\rho_c$
$B^{\alpha}{}_{\alpha} = 0$	$\nabla.\vec{B} = 0$
$E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^{\gamma}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\beta} - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}$	$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi\vec{j}$

($j^{\alpha} \equiv$ Charge current 4-vector; $\rho_c = -j^{\alpha} U_{\alpha} \equiv$ charge density — as measured by the observer with 4-velocity U^{α})

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- Decomposing: $F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}B^{\mu}{}_{\gamma}U^{\sigma}$, we see that Maxwell's equations indeed involve only tidal tensors and sources.

The Gravitational analogue of Maxwell's Equations

By performing, on the gravitational tidal tensors, the same operations that led to Maxwell's equations, i.e, taking the traces and antisymmetric parts, we obtain the analogous set of equations:

The Gravitational analogue of Maxwell's Equations

Electromagnetism Maxwell Equations	Gravity Eqs. Grav. Tidal Tensors
$E^{\alpha}{}_{\alpha} = 4\pi\rho_c$	$\mathbb{E}^{\alpha}{}_{\alpha} = 4\pi\left(2\rho_m + T^{\alpha}{}_{\alpha}\right)$
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- Strikingly similar when the setups are stationary in the observer's rest frame. Otherwise, tell us that gravitational and electromagnetic interactions differ significantly, since the tidal tensors do not have the same symmetries.

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- *Charges*: the gravitational analogue of ρ_c is $2\rho_m + T^{\alpha}_{\alpha}$ ($\rho_m + 3p$ for a perfect fluid) $\Rightarrow$ in gravity, pressure and all material stresses contribute as sources.

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- *Ampére law*: in stationary (in the observer's rest frame) setups, equations $B_{[\alpha\beta]}$ and $\mathbb{H}_{[\alpha\beta]}$ match up to a factor of 2 $\Rightarrow$ currents of mass/energy source gravitomagnetism like currents of charge source magnetism.

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- *Absence of electromagnetic-like induction effects in gravity*:
  - $\mathbb{E}_{\mu\gamma}$ always symmetric $\Rightarrow$ no gravitational analogue to Faraday's law of induction!

The Gravitational analogue of Maxwell's Equations

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- *Absence of electromagnetic-like induction effects in gravity*:
  - Induction term $\star F_{\alpha\beta;\gamma}U^{\gamma}$ in $B_{[\alpha\beta]}$ has no counterpart in $\mathbb{H}_{[\alpha\beta]}$ $\Rightarrow$ no gravitational analogue to the magnetic fields induced by time varying electric fields.

Electromagnetism Maxwell Field Equations	Gravity Einstein Field Equations
$F^{\alpha\beta}{}_{;\beta} = J^\beta$	$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha\right)$
•Time Projection:	•Time-Time Projection:
$E^\alpha{}_\alpha = 4\pi\rho_c$	$\mathbb{E}^\alpha{}_\alpha = 4\pi\left(2\rho_m + T^\alpha{}_\alpha\right)$
•Space Projection:	•Time-Space Projection:
$B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^\beta - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma$	$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma$
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- Space-Space part of Einstein's equations (involving the tensor $\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\mu\beta\nu} U^\mu U^\nu$) has no electromagnetic analogue.

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- By replacing $\{E_{\alpha\beta},\, B_{\alpha\beta}\} \leftrightarrow \{\mathbb{E}_{\alpha\beta},\, \mathbb{H}_{\alpha\beta}\}$ in Maxwell equations, one *almost* obtains some of Einstein's equations!

Comparison with Linear approach to GEM

Eqs. Grav. Tidal Tensors (exact, covariant)	Linearized theory (non-covariant)
$\mathbb{E}^{\alpha}{}_{\alpha} = 4\pi\left(2\rho_m + T^{\alpha}_{\ \alpha}\right)$	$\nabla.\vec{E}_G = 4\pi\rho_m$
$\mathbb{H}^{\alpha}{}_{\alpha} = 0$	$\nabla.\vec{B}_G = 0$
$\mathbb{E}_{[\alpha\beta]} = 0$	$\nabla \times \vec{E}_G = 0$
$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^{\sigma}U^{\gamma}$	$\frac{1}{2}\nabla \times \vec{B}_G = 4\pi\vec{J}$

- Equations for Gravitational Tidal Tensors are a generalization, for arbitrary fields and frames, of the "gravitoelectromagnetic" equations derived in literature (eg. Thorne et al., 1977; Harris, 1991; Ciufolini & Wheeler, 1995; Ruggiero & Tartaglia, 2002).

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- Physical analogy based on linear GEM is restricted to stationary phenomena (sheds light on ongoing debate).

Matching Between Tidal Tensors

The analogy based on tidal tensors does not rely on a similarity between the tidal tensors.

- Despite playing analogous roles in dynamics, gravitational and electromagnetic tidal tensors are generically very different:
  - gravitational tidal tensors are non-linear, spatial and symmetric (in vacuum, in the magnetic case);
  - electromagnetic tidal tensors are linear and generically non-symmetric and non-spatial.

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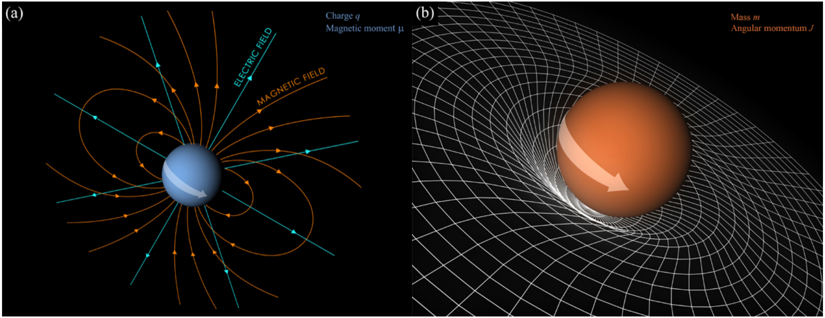
Nevertheless, a matching occurs under certain special conditions.

Spinning Charge vs Spinning Mass

Consider the elementary example of analogous physical problems:

- the metric outside a rotating spherical mass (which is asymptotically described by the Kerr solution);
- the electromagnetic field produced by a rotating charged sphere (in Minkowski spacetime).



Spinning Charge vs Spinning Mass

Far field limit $r \to \infty$; observer at rest with respect to the center of the spheres.

- Spinning Charge — Electromagnetic tidal tensors

$$E_{\alpha\beta}dx^\alpha dx^\beta = -\frac{2q}{r^3}dr^2 + \frac{q}{r}d\Omega_2$$

$$B_{\alpha\beta}dx^\alpha dx^\beta = \frac{q}{2m}\frac{3J\cos\theta}{r^2}\left(d\Omega_2 - \frac{2}{r^2}dr^2 - \frac{2\tan\theta}{r}drd\theta\right)$$

- Spinning Mass — Gravitational tidal tensors

$$\mathbb{E}_{\alpha\beta}dx^\alpha dx^\beta \simeq -\frac{2m}{r^3}dr^2 + \frac{m}{r}d\Omega_2$$

$$\mathbb{H}_{\alpha\beta}dx^\alpha dx^\beta \simeq \frac{3J\cos\theta}{r^2}\left(d\Omega_2 - \frac{2}{r^2}dr^2 - \frac{2\tan\theta}{r}drd\theta\right)$$

- Identifying $m \leftrightarrow q$: electric tidal tensors asymptotically match; magnetic tidal tensors match up to a factor of 2.

Linearized gravitational perturbations

The matching occurs because far from the source, the non-linearities of the gravitational field are negligible. These results are a special case of a more general principle.

- Consider arbitrary <u>stationary</u> gravitational perturbations:

$$ds^2 = -(1-2\Phi)\,dt^2 - 4\mathcal{A}_j dt dx^j + (\hat{g}_{ij} + \Theta_{ij})\,dx^i dx^j$$

and an arbitrary electromagnetic field in Minkowski spacetime:

$$A^\alpha = (\phi, \vec{A}); \quad ds^2 = -dt^2 + \hat{g}_{ij} dx^i dx^j$$

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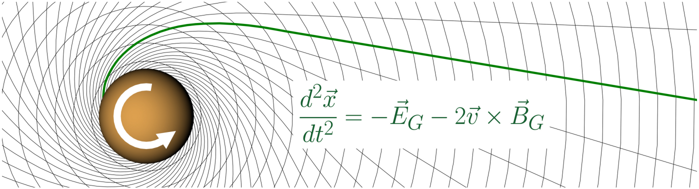
- For a *static* observer $U^\alpha = \delta^\alpha_0$, the *linearized* gravitational tidal tensors are the same, identifying $(\phi, A^i) \leftrightarrow (\Phi, \mathcal{A}^i)$, as the electromagnetic ones:

$$\mathbb{E}_{ij} \simeq -\hat{D}_i \hat{D}_j \phi = E_{ij}, \quad \mathbb{H}_{ij} \simeq \hat{\epsilon}_{ilk} \hat{D}_j \hat{D}^l A^k = B_{ij}$$

  ($\hat{D}$ denotes covariant derivative with respect to $\hat{g}_{ij}$)

Linearized gravitational perturbations

- For a static observer, gravity would seem consisting of an electric field $\vec{E}_G \equiv -\nabla\phi$ and a magnetic field $\vec{B}_G \equiv \nabla \times \vec{A}$.
  - Test particles will move on geodesics described (to first order in the velocity) by an equation analogous to Lorentz force law:



  - and a gyroscope will precess like a magnetic dipole under a torque $\tau = -\vec{S} \times \vec{B}_G$ — which is the *Lense-Thirring Effect*, measured to a 10% accuracy by Ciufolini-Pavlis analysis of LAGEOS data, and subject of experimental scrutiny by NASA/Stanford Gravity Probe B and the upcoming LARES missions.

Linearized gravitational perturbations

- However, when the fields are not stationary in the observer's rest frame — that is the case of a non-stationary metric, or an observer moving in a stationary metric — the GR tidal tensors are very different from the EM ones, so that the physical analogy $(\phi, A^i) \leftrightarrow (\Phi, \mathcal{A}^i)$ no longer holds.

Linearized gravitational perturbations

- Time-dep. setups, and observer with 4-velocity $U^\alpha = (u^0, u^i)$

$$
\begin{aligned}
E_{00} &= (\dot{\phi}_{;i} + \ddot{A}_i)u^i \\
E_{0i} &= (\phi_{;ki} + \dot{A}_{k;i})u^k \\
E_{i0} &= -(\dot{\phi}_{;i} + \ddot{A}_i)u^0 + 2\dot{A}_{[j;i]}u^j \\
E_{ij} &= -(\phi_{;ij} + \dot{A}_{i;j})u^0 + 2A_{[k;i]j}u^k
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_{00} &\simeq -\left(\Phi_{;ji} + 2\dot{\mathcal{A}}_{j;i} + \ddot{\Theta}_{ij}\right)u^i u^j \\
\mathbb{E}_{0i} = \mathbb{E}_{i0} &\simeq \left(\Phi_{;ij} + 2\dot{\mathcal{A}}_{(i;j)} + \ddot{\Theta}_{ij}\right)u^0 u^j + 2\left(\dot{\Theta}_{j[i;k]} - \mathcal{A}_{[k;i]j}\right)u^k u^j \\
\mathbb{E}_{ij} = \mathbb{E}_{ji} &\simeq 2\left(\dot{\Theta}_{k(i;j)} - \dot{\Theta}_{ij;k} + \mathcal{A}_{k;ij} - \mathcal{A}_{(i;j)k}\right)u^0 u^k \\
&\quad + \left(2\Theta_{l(i;j)k}u^k u^l - \Theta_{ij;lk} - \Theta_{lk;ij}\right)u^k u^l \\
&\quad - \left(\Phi_{;ij} + 2\dot{\mathcal{A}}_{(i;j)} + \ddot{\Theta}_{ij}\right)(u^0)^2
\end{aligned}
$$

Linearized gravitational perturbations

- Time-dep. setups, and observer with 4-velocity $U^{\alpha} = (u^0, u^i)$

$$
\begin{aligned}
B_{00} &= -\hat{\epsilon}_{ijk}\dot{A}^{j;i}u^k \\
B_{0i} &= -\hat{\epsilon}_{ljk}A^{j;l}{}_{;i}u^k \\
B_{i0} &= \hat{\epsilon}_{kji}\dot{A}^{j;k}u^0 + \hat{\epsilon}_{jik}\left(\dot{\phi}^{;j} + \ddot{A}^j\right)u^k \\
B_{ij} &= \hat{\epsilon}_i{}^{lm}A_{m;lj}u^0 + \hat{\epsilon}_{lik}\left(\phi^{;l}{}_{;j} + \dot{A}^l{}_{;j}\right)u^k
\end{aligned}
$$

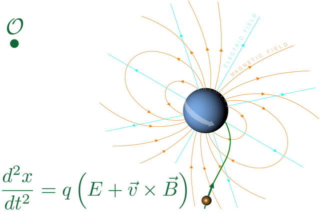
$$
\begin{aligned}
\mathbb{H}_{00} &\simeq \hat{\epsilon}_{imn}\left(\mathcal{A}^{n;m}{}_{;j} + \dot{\Theta}_j{}^{n;m}\right)u^i u^j \\
\mathbb{H}_{i0} &\simeq \hat{\epsilon}_i{}^{lk}\left(\dot{\Theta}_{jk;l} - \mathcal{A}_{k;lj}\right)u^0 u^j - \hat{\epsilon}_{ik}{}^{l}\left(\Phi_{;jl} + 2\dot{\mathcal{A}}_{(l;j)} + \ddot{\Theta}_{lj}\right)u^k u^j \\
\mathbb{H}_{0i} &\simeq \hat{\epsilon}_j{}^{lk}\left(\dot{\Theta}_{ik;l} - \mathcal{A}_{k;li}\right)u^0 u^j - \hat{\epsilon}_j{}^{lm}\left(\Theta_{l[k;i]m} + \Theta_{m[i;k]l}\right)u^j u^k \\
\mathbb{H}_{ij} &\simeq \hat{\epsilon}_i{}^{lk}\left(\mathcal{A}_{k;lj} + \dot{\Theta}_{jk;l}\right)(u^0)^2
\end{aligned}
$$

Linearized gravitational perturbations

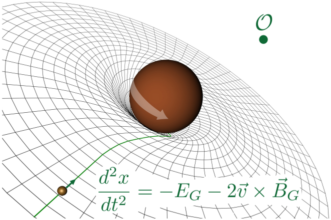
Relative acceleration of two particles (initially) at rest:

Electromagnetism	Gravity
$\frac{D^2\delta x^i}{d\tau^2} = \frac{q}{m}E_{ij}\delta x^j$	$\frac{D^2\delta x^i}{d\tau^2} = -\mathbb{E}_{ij}\delta x^j$
$= -\frac{q}{m}(\phi_{;ij} + \dot{A}_{i;j})\delta x^j$	$= -\left(\Phi_{;ij} + 2\dot{\mathcal{A}}_{(i;j)} + \ddot{\Theta}_{ij}\right)\delta x^j$
$\equiv \frac{q}{m}E_{i;j}\delta x^j$	

Linearized gravitational perturbations

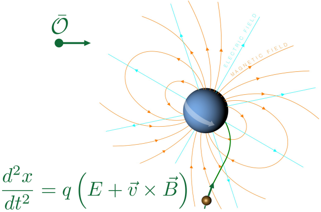


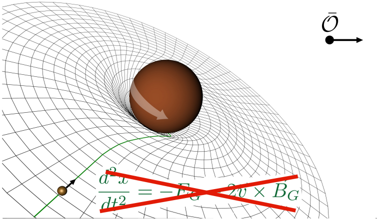
$$\frac{d^2x}{dt^2} = q\left(E + \vec{v}\times\vec{B}\right)$$



$$\frac{d^2x}{dt^2} = -E_G - 2\vec{v}\times\vec{B}_G$$

Linearized gravitational perturbations

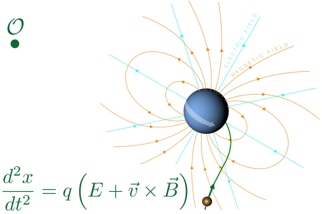


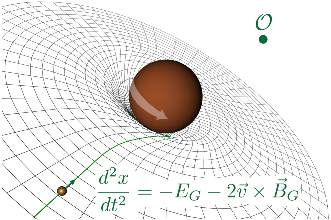


$$\frac{d^2\vec{x}}{dt^2} = \nabla\Phi + 2\frac{\partial\vec{\mathcal{A}}}{\partial t} - 2\vec{v}\times\nabla\times\vec{\mathcal{A}} - \frac{\partial\Phi}{\partial t}\vec{v} - \frac{2\partial\Theta_i{}^j}{\partial t}v^i\vec{e}_j$$

$$\text{EM Lorentz Force: } \frac{d^2\vec{x}}{dt^2} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times\nabla\times\vec{A}$$

Linearized gravitational perturbations

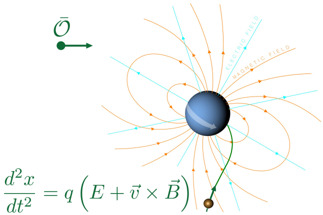


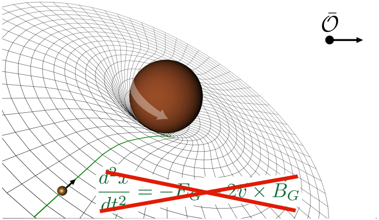


Force on gyroscope: $\vec{F}_G = -\nabla(\vec{B}_G.\vec{S})$

Force on dipole: $\vec{F}_{EM} = \frac{q}{2m}\nabla(\vec{B}.\vec{S})$

Linearized gravitational perturbations





Force on gyroscope: $\vec{F}_G = -\nabla(\vec{B}_G.\vec{S}) + \hat{\epsilon}_i{}^{lk}\dot{\Theta}^j{}_{k;l}\vec{e}_j$

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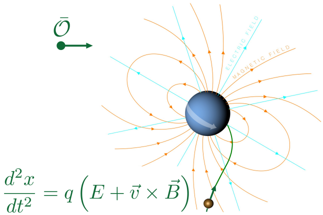

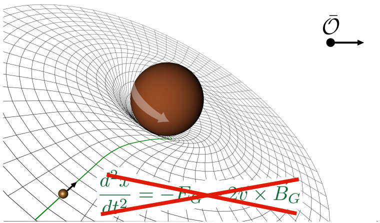

$$\text{Gyroscope precession: } \frac{d\vec{S}}{d\tau} = -\vec{S} \times \vec{B}_G - \frac{\partial \Theta^{ij}}{\partial t} S_j \vec{e}_i$$

Ultra-stationary spacetimes

Ultra-stationary spacetimes are a special class of stationary spacetimes whose metric has a constant g_{00} component in the chart where it is explicitly time independent.

Ultra-stationary spacetimes

The line element is, generically:

$$ds^2 = -\left(dt + A_i(x^k)dx^i\right)^2 + \hat{g}_{ij}(x^k)dx^i dx^j$$

In these spacetimes (Herdeiro & Gibbons, 1999), the Klein-Gordon equation:

$$\Box\Psi = m^2\Psi\,,$$

with the ansatz $\Psi = e^{-iEt}\psi(x^j)$, reduces to a time independent Schrödinger equation: $H\psi = \epsilon\psi$, where

$$H = \frac{(\vec{P} + E\vec{A})^2}{2m}\ , \qquad \epsilon = \frac{E^2 - m^2}{2m}$$

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$$H = \frac{(\vec{P} + E\vec{A})^2}{2m}\ , \quad \epsilon = \frac{E^2 - m^2}{2m}$$

- This is the non-relativistic problem of a particle with "charge" $-E$ and mass m, living in a curved 3-space with metric $\hat{g}_{ij}$, under the action of a magnetic field $\vec{B} = \nabla \times \vec{A}$.

Ultra-stationary spacetimes

The covariant derivative of the magnetic field $\vec{B}$ turns out to be, up to the usual factor of 2, the same as the *exact* gravito-magnetic tidal tensor!

$$\hat{D}_j B_i = B_{ij} = \hat{\epsilon}_{lmi} \hat{D}_j \hat{D}^l A^m = 2\mathbb{H}_{ij}$$

($\hat{D}$ denotes covariant derivatives with respect to $\hat{g}_{ij}$)

- This is a highly non-trivial realization of the analogy, since there is an *exact* matching between tidal tensors from a linear theory (electromagnetism) with the ones from a non-linear theory (gravity);
- Provides valuable insight for the understanding of some properties of these spacetimes.

Ultra-stationary spacetimes - The Godel Universe

"*Einstein told me, that Godel's papers were the most important ones on relativity since his own original paper appeared*" (Oskar Morgenstern, 1972)

The Godel Universe is an exact solution of Einstein's equations for the case of a homogeneous universe filled with a perfect, pressureless, rotating fluid.

Ultra-stationary spacetimes - The Godel Universe

- Portrayed in literature as a homogeneous rotating universe - a conceptually hard definition since it means that it rotates around *every* point (eg. Ciufolini, 1995);

- The vanishing magnetic part of the Weyl tensor has also led to some conceptual difficulties:
  - "*The Godel metric, as with the Davidson metric, is a counterexample to the hypothesis that rotation is the source of the magnetic part of the Weyl tensor.*" (C. Lozanovski, C.B.G. McIntosh, 1999)
  - "*the gravito-magnetic field $\mathcal{H}_{ab}$ has the source $(\rho + p)\omega_a$, which we identify as a gravito-magnetic 'charge' density. Note, however, that angular momentum density does not always generate a gravito-magnetic field. The Godel solution provides a counter-example* (Roy Maartens, Bruce A. Basset, 1998)

Ultra-stationary spacetimes - The Godel Universe

Within the analogy proposed herein, both these facts have a straightforward interpretation. The line element is given by:

$$ds^2 = -\left(dt + A_y dy\right)^2 + \hat{g}_{ij} dx^i dx^j$$

with:

$$A_y = e^{\sqrt{2}\omega x}$$

$$[\hat{g}_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}e^{2\sqrt{2}\omega x} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ultra-stationary spacetimes - The Godel Universe

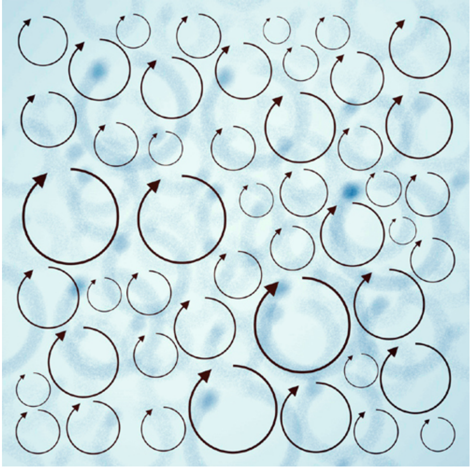
The Klein-Gordon equation maps this metric into the magnetic field $\vec{B} = 2\omega\vec{e}_z$ living in the three-space of metric:

$$[\hat{g}_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}e^{2\sqrt{2}\omega x} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $B_{ij} = \hat{D}_j B_i = 0 \Rightarrow$ the magnetic field is uniform in this 3-space
- thus, the physical interpretation for the vanishing of the magnetic part $\mathbb{H}_{\alpha\beta}$ of the Riemann (and hence Weyl) tensor is that these tensors are magnetic tidal tensors, and hence they must vanish since the Gödel universe has a *uniform* gravitomagnetic field.

Ultra-stationary spacetimes - The Godel Universe

- The concept of homogeneous rotation is then easily assimilated by an analogy with the more familiar picture of a gas of charged particles subject to a uniform magnetic field: there are Larmor orbits around any point.



Magnetic dipole vs Gyroscope

Electromagnetic Force on a Magnetic Dipole	Gravitational Force on a Spinning Particle
$F_{EM}^{\beta} = \frac{q}{2m} B^{\alpha\beta} S_{\alpha}$	$F_{G}^{\beta} = -\mathbb{H}^{\alpha\beta} S_{\alpha}$

The explicit analogy between F_{EM}^{β} and F_{G}^{β} is ideally suited to:

- Compare the two interactions: amounts to compare $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, which is crystal clear from the equations for gravitational and electromagnetic tidal tensors;
- Visualize, in analogy with the more familiar electromagnetic ones, gravitational effects which are not transparent in the Papapetrou's original form.

Magnetic dipole vs Gyroscope

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Example: non-geodesic motion of a gyroscope (even) in the absence of rotating sources (e.g. Schwarzschild spacetime) — an effect readily visualized with the help of the explicit analogy:

- it is the magnetic tidal tensor as seen by the dipole/gyroscope what determines the force exerted upon it;
  - thus, a gyroscope (in non-radial motion) deviates from geodesic motion in Schwarzschild spacetime by the same reason that a magnetic dipole will suffer a force even in the coulomb field of a point charge: in its "rest" frame, there is a non-vanishing magnetic tidal tensor.

Magnetic dipole vs Gyroscope

- An effect not explained in the first order estimate derived in the framework of the gravito-electromagnetic analogy known from linearized theory (e.g. Wald, 1972; Harris, 1991):

$$\vec{F}_G = -\nabla(\vec{S} \cdot \vec{B}_G)$$

similar to the 3-D non-covariant form for the electromagnetic force on a dipole:

$$\vec{F}_{EM} = \nabla(\vec{\mu}.\vec{B})$$

- as is clear in the tidal tensor formalism, such expression can be valid only if the gyroscope is at *rest* in a *stationary*, besides weak, gravitational field $\Rightarrow$ therefore not suited to describe motion.

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- accounts only for the coupling between the spin of the source and the spin of the gyroscope, hiding the fact that a force acts on the gyroscope even in the absence of rotating sources (for example, in Schwarzschild spacetime).

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similar to the 3-D non-covariant form for the electromagnetic force on a dipole:

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- Important pysical content lost in these 3-D expressions is unveiled by the time components of F^{α}_{EM} and F^{α}_{G}.

Time projection of F^{α}_{EM} in the dipole's proper frame:

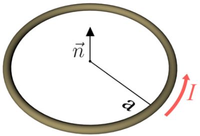
$$F^{\alpha}_{EM} U_{\alpha} = -\frac{dm}{d\tau}$$

- $m = P^{\alpha} U_{\alpha} \equiv$ proper mass of the test particle
-

Time projection of F^{α}_{EM} in the dipole's proper frame:

$$F^{\alpha}_{EM} U_{\alpha} = -\frac{dm}{d\tau} = \frac{\partial \vec{B}}{\partial t} . \vec{\mu}$$

- The magnetic dipole may be thought as a small current loop.



(Area of the loop
$A = 4\pi a^2$; $I \equiv$ current through the loop, $\vec{n} \equiv$unit vector normal to the loop)

- The magnetic dipole moment is given by $\vec{\mu} = IA\vec{n}$

Time projection of F^{α}_{EM} in the dipole's proper frame:

$$F^{\alpha}_{EM}U_{\alpha} = -\frac{dm}{d\tau} = \frac{\partial \vec{B}}{\partial t}.\vec{\mu} = \frac{\partial \vec{B}}{\partial t}.\vec{n}AI = \frac{\partial \Phi}{\partial t}I$$

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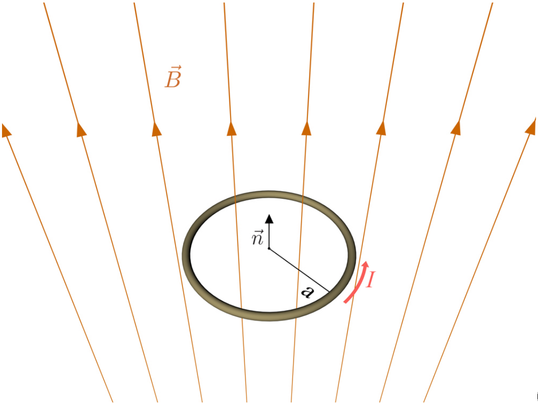


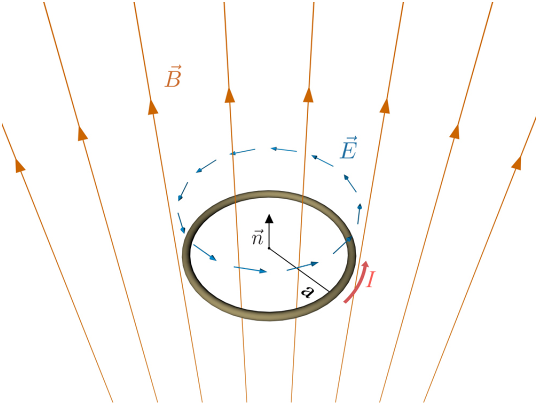
(Area of the loop
$A = 4\pi a^2$; $I \equiv$ current through the loop, $\vec{n} \equiv$unit vector normal to the loop)

- $\vec{B}A\vec{n} = \Phi \equiv$ magnetic flux trough the loop

Therefore, by Faraday´s law of induction:

$$F^{\alpha}_{EM}U_{\alpha} = -\frac{dm}{d\tau} = \frac{\partial \vec{B}}{\partial t}.\vec{\mu} = \frac{\partial \vec{B}}{\partial t}.\vec{n}AI = \frac{\partial \Phi}{\partial t}I = -I\oint_{loop} \vec{E}.\vec{ds}$$

- $\vec{E} \equiv$ Induced electric field



- Hence $F^{\alpha}_{EM}U_{\alpha} = -dm/d\tau$ is minus the power transferred to the dipole by Faraday's law of induction.

Time Projection of F_G^α — no gravitational induction

Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have*

$$F_G^\alpha U_\alpha = -\frac{dm}{d\tau} = 0$$

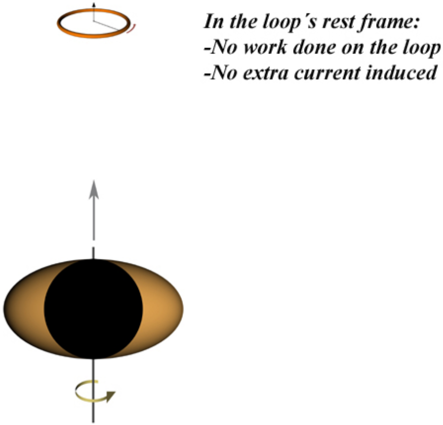
- No work is done by induction $\Rightarrow$ the proper mass of the gyroscope is constant.
- Spatial character of gravitational tidal tensors <u>precludes</u> induction effects analogous to the electromagnetic ones.

Time Projection of F_G^{α} — no gravitational induction

Example: A mass loop subject to the time-varying "gravitomagnetic field" of a moving Kerr Black Hole

Time Projection of F_G^α — no gravitational induction

Example: A mass loop subject to the time-varying "gravitomagnetic field" of a moving Kerr Black Hole



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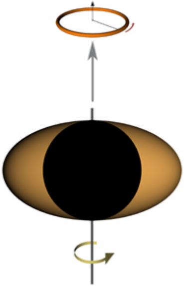

In the loop's rest frame:
-No work done on the loop
-No extra current induced

Time components on arbitrary frames

- Electromagnetism:
  In an arbitrary frame, in which the dipole has 4-velocity $U^\beta = \gamma(1, \vec{v})$, the time component of the force exerted on a magnetic dipole is:

$$
\begin{aligned}
(F_{EM})_0 &\equiv -\frac{DE}{d\tau} = \frac{F^\beta_{EM} U_\beta}{\gamma} - F^i_{EM} v_i = -\left(\frac{1}{\gamma}\frac{dm}{d\tau} + F^i_{EM} v_i\right) \\
&\equiv -(\mathcal{P}_{mech} + \mathcal{P}_{ind})
\end{aligned}
$$

  where $E \equiv -P_0$ is the energy of the dipole and we identify:

  - $\mathcal{P}_{ind} = \frac{1}{\gamma}\frac{dm}{d\tau} = -F^\beta_{EM} U_\beta / \gamma \equiv$ induced power
  - $\mathcal{P}_{mech} = F^i_{EM} v_i$ "mechanical" power transferred to the dipole by the 3-force F^i_{EM} exerted upon it.

Time components on arbitrary frames

- Gravity:
Since $F_G^\alpha U_\alpha = -dm/d\tau = 0$, we have

$$(F_G)_0 = -\frac{DE}{d\tau} = -F_G^i v_i$$

Static Observers — Electromagnetism

- When the fields are stationary in the observer's rest frame, $(F_{EM})_0 = 0 \Rightarrow$ no work is done on the magnetic dipole.
  - Related to a basic principle from electromagnetism: the total amount of **work done by a static magnetic field on an arbitrary system of currents is zero.**

Static Observers — Electromagnetism

- When the fields are stationary in the observer's rest frame, $(F_{EM})_0 = 0 \Rightarrow$ no work is done on the magnetic dipole.
  - $\mathcal{P}_{mech}$ and $\mathcal{P}_{ind}$ exactly cancel out

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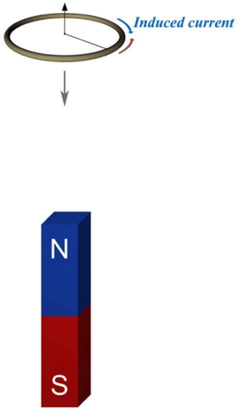

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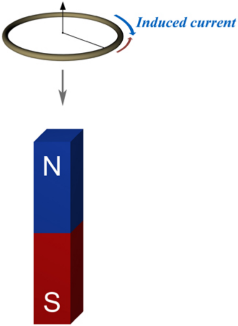


Static Observers — Gravity

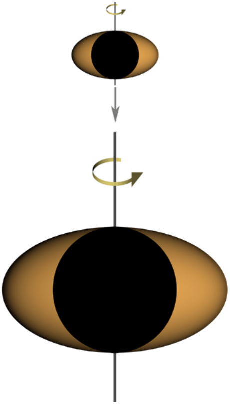
- In gravity, since those induction effects are absent, such cancellation does not occur:

$$(F_G)_0 = -\frac{DE}{d\tau} = -F^i_G v_i \neq 0$$

- Therefore, the stationary observer must measure a non-zero work done on the gyroscope.
  - That is to say, **a static "gravitomagnetic field" (unlike its electromagnetic counterpart) does work.**
  - And there is a known consequence of this fact: the spin dependent upper bound for the energy released when two black holes collide, obtained by Hawking (1971) from the area law.
  - For the case with spins aligned, from Hawking's expression one can infer a gravitational spin-spin interaction energy (Wald, 1972).

Static Observers — Gravitational spin interaction

Take the gyroscope to be a small Kerr black hole of spin $S \equiv \sqrt{S^\alpha S_\alpha}$ falling along the symmetry axis of a larger Kerr black hole of mass m and angular momentum $J = am$.



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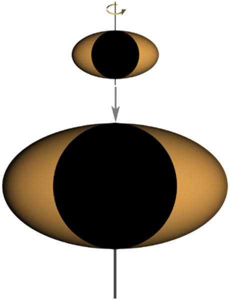

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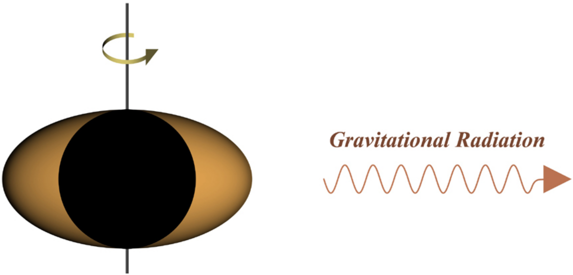

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- The time component of the force acting on the small black hole is given by:

$$(F_G)_0 \equiv \frac{DP_0}{D\tau} = -\frac{dE}{d\tau} = -\frac{2ma\left(3r^2 - a^2\right) U^r S}{\left(r^2 + a^2\right)^3}$$

Integrating this equation from infinity to the horizon one obtains

$$\int_\infty^{r_+} (F_G)_0 \equiv \Delta E = \frac{aS}{2m\left[m + \sqrt{m^2 - a^2}\right]} \, ,$$

which is precisely Hawking's spin-spin interaction energy for this particular setup.

Electromagnetism	Gravity
Worldline deviation: $\frac{D^2\delta x^\alpha}{D\tau^2} = \frac{q}{m} E^\alpha{}_\beta \delta x^\beta$	Geodesic deviation: $\frac{D^2\delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha{}_\beta \delta x^\beta$
Force on magnetic dipole: $\frac{DP^\beta}{D\tau} = \frac{q}{2m} B^{\alpha\beta} S_\alpha$	Force on gyroscope: $\frac{DP^\beta}{D\tau} = -\mathbb{H}^{\alpha\beta} S_\alpha$
Maxwell Equations:	Eqs. Grav. Tidal Tensors:
$E^\alpha{}_\alpha = 4\pi\rho_c$	$\mathbb{E}^\alpha{}_\alpha = 4\pi\left(2\rho_m + T^\alpha{}_\alpha\right)$
$E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^\gamma$	$\mathbb{E}_{[\alpha\beta]} = 0$
$B^\alpha{}_\alpha = 0$	$\mathbb{H}^\alpha{}_\alpha = 0$
$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\beta - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma$	$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma$

Conclusions

- The tidal tensor formalism makes transparent both the similarities and key differences between the gravitational and electromagnetic interactions.
- At the same time it unveils a new physical analogy between General relativity and electromagnetism, based on exact, covariant, and fully general equations.
- The non-geodesic motion of a spinning test particle not only can be easily understood, but also exactly described, by a simple application of the analogy based on tidal tensors.
- An unification within gravito-electromagnetism was also achieved: the (exact) connection between ultra-stationary spacetimes and magnetic fields in some curved manifolds was seen to originate from the same fundamental principle as the analogy from linearized theory: a matching between tidal tensors.

- Issues concerning previous approaches in literature were clarified — namely, the limit of validity of the usual linear approach to gravitoelectromagnetism, and the physical interpretation of the magnetic parts of the Riemann/Weyl tensors.

Acknowledgments

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Thank you