Space-time topography:

studying Riemannian geometry by means of freely falling clocks

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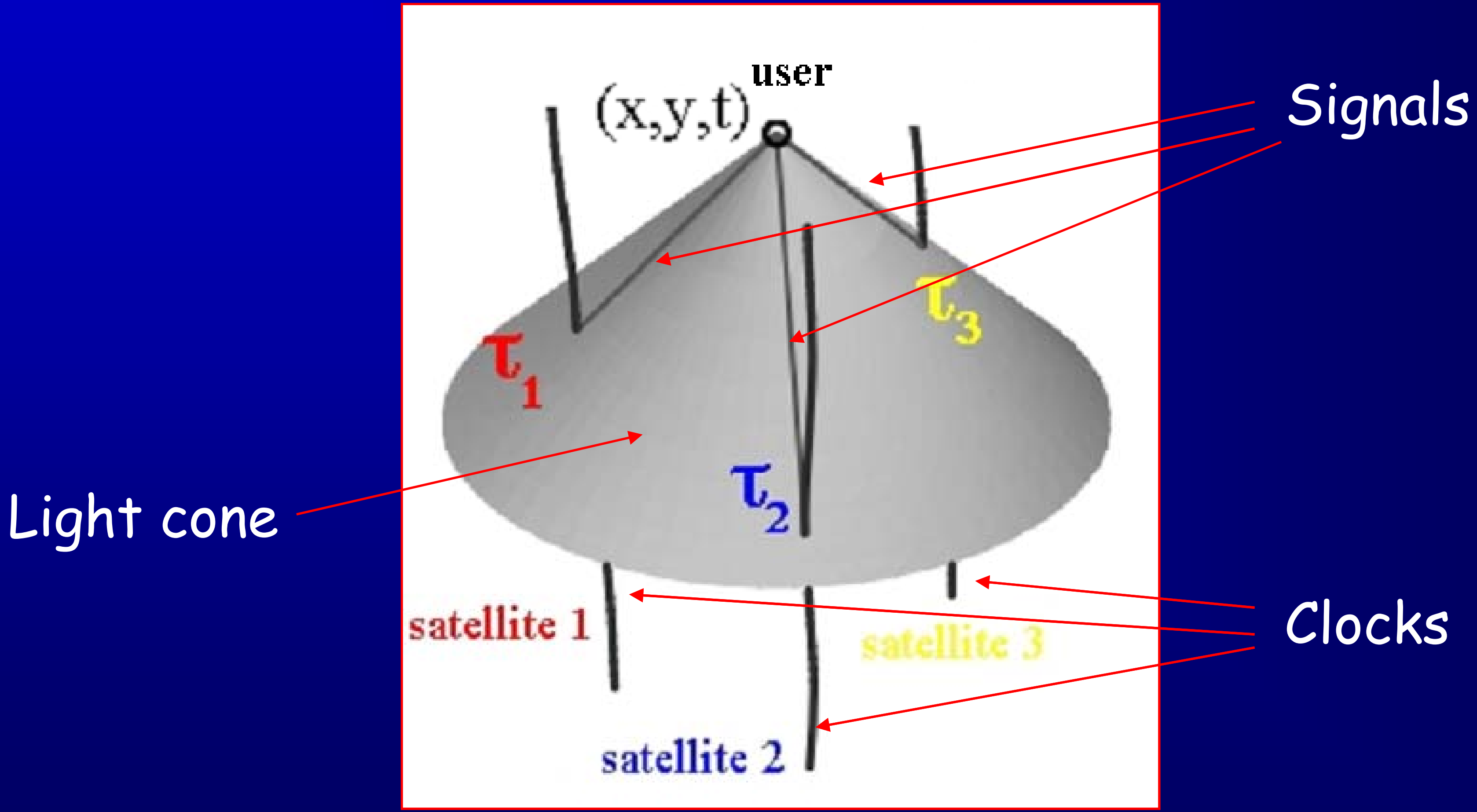
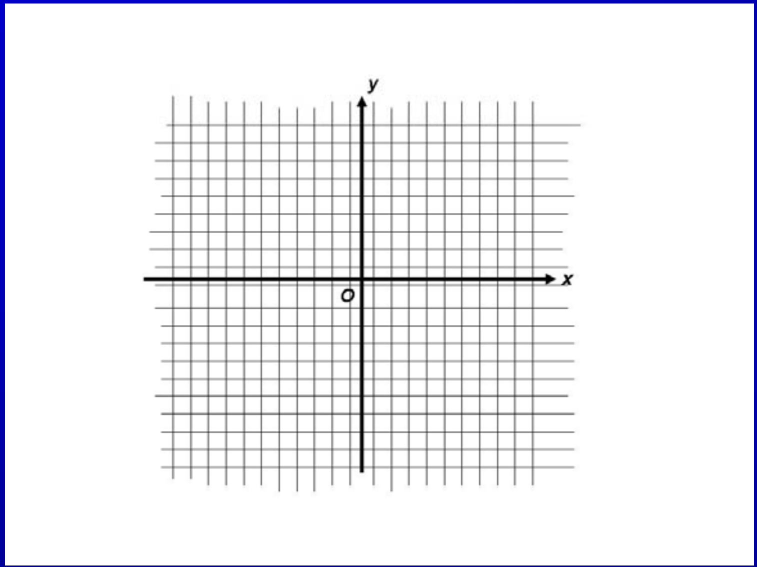
Coordinates and positioning

- Space-time is a 4-dimensional generally curved metric manifold
- Gaussian coordinates may be used to localize events on the manifold.

Emission coordinates

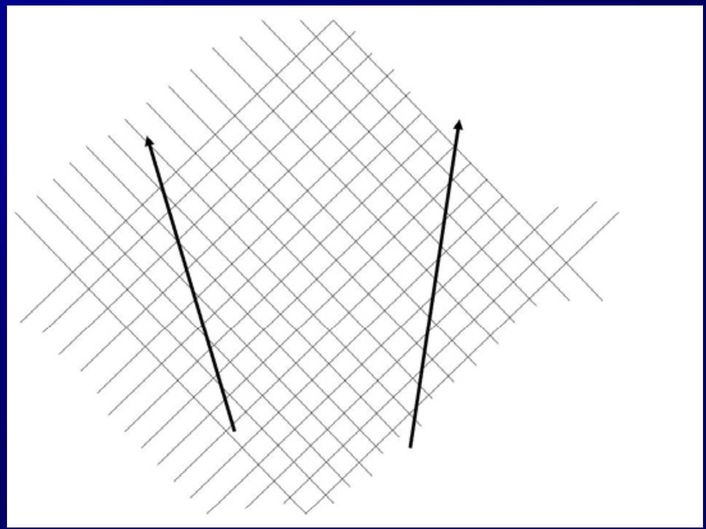


Null or light coordinates

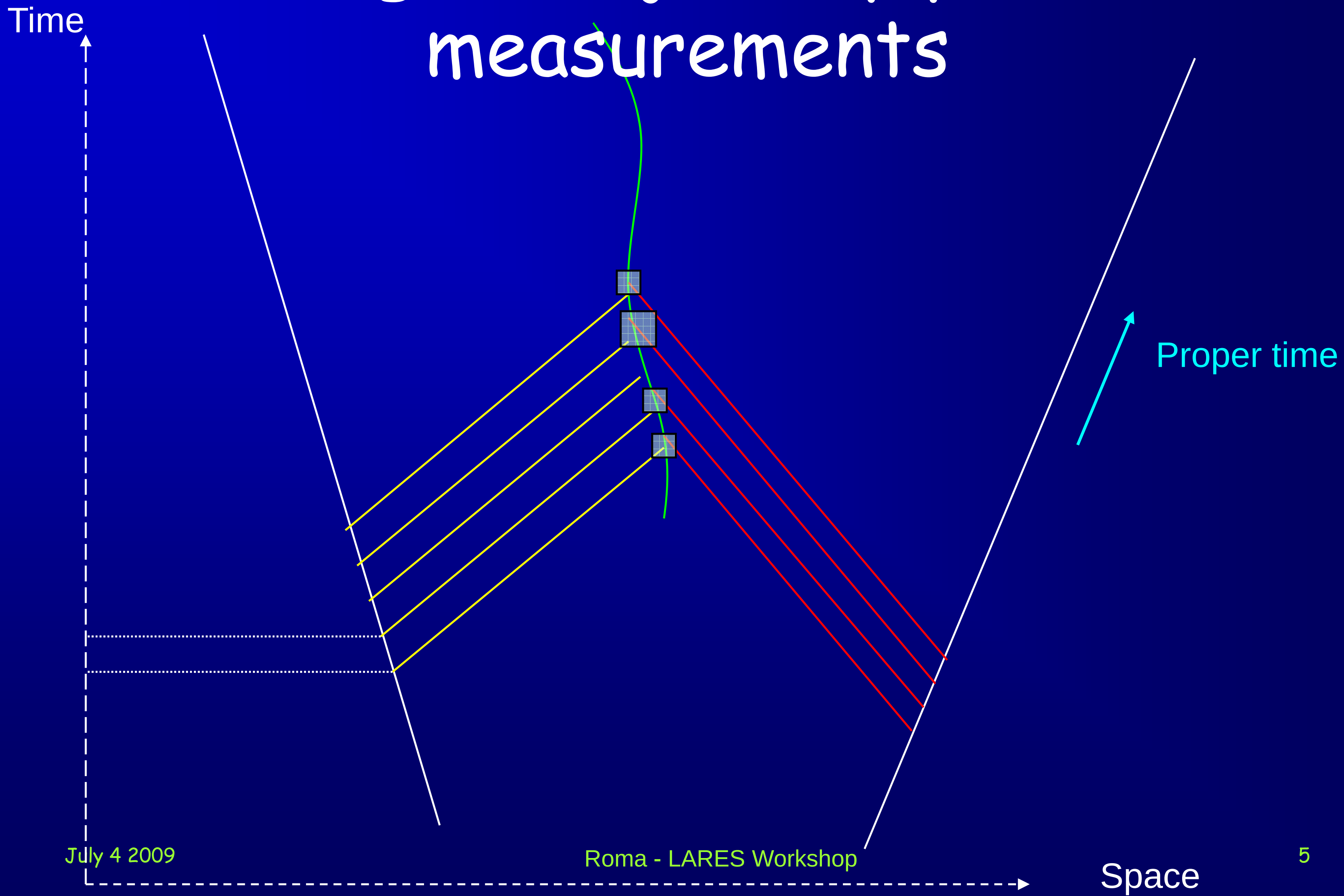


Cartesian grid

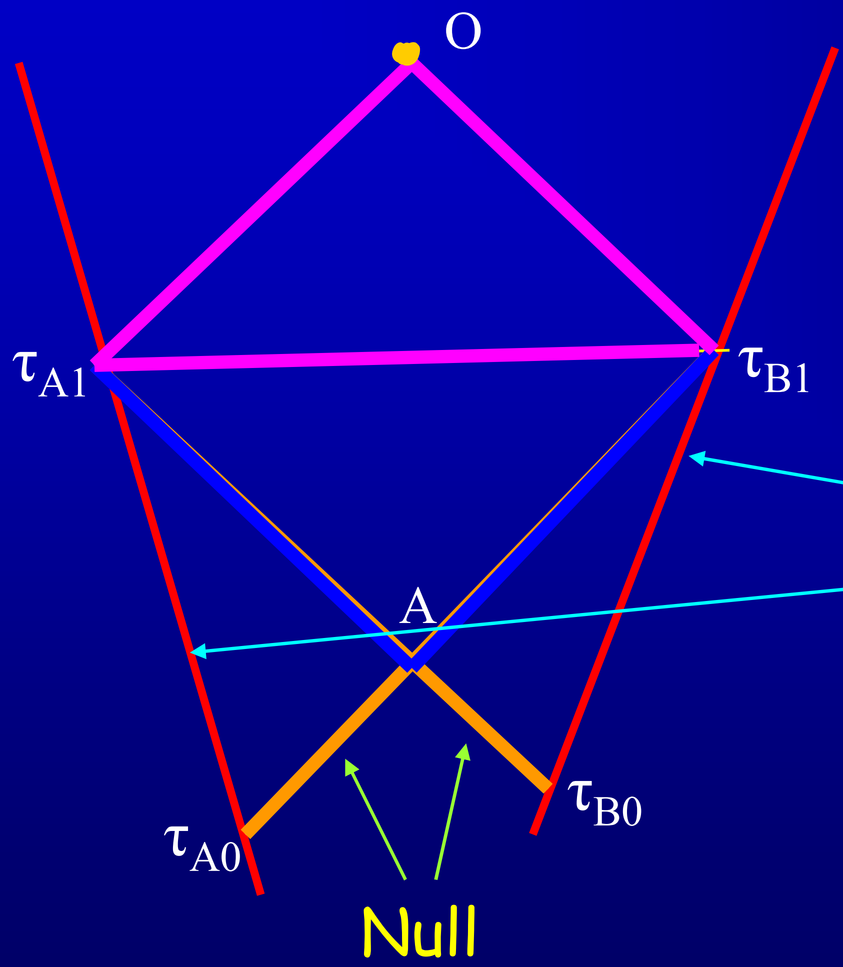
Light rays grid



Tracking an object by pure time measurements



Positioning



Null-null triangles

These are measured

Light cone equations with 4 satellites (clocks)

$$(T-\gamma_a\tau_a)^2 = (X-x_a)^2 + (Y-y_a)^2 + (Z-z_a)^2$$
$$(T-\gamma_b\tau_b)^2 = (X-x_b)^2 + (Y-y_b)^2 + (Z-z_b)^2$$
$$(T-\gamma_c\tau_c)^2 = (X-x_c)^2 + (Y-y_c)^2 + (Z-z_c)^2$$
$$(T-\gamma_d\tau_d)^2 = (X-x_d)^2 + (Y-y_d)^2 + (Z-z_d)^2$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad x = x_0 + \xi\tau \qquad \beta^2 = \xi^2 + \eta^2 + \zeta^2 < 1$$

Equivalent system

$$A_1T - B_1X - C_1Y - D_1Z = N_1$$
$$A_2T - B_2X - C_2Y - D_2Z = N_2$$
$$A_3T - B_3X - C_3Y - D_3Z = N_3$$
$$\left(T - \gamma_a t_a\right)^2 = \left(X - x_a\right)^2 + \left(Y - y_a\right)^2 + \left(Z - z_a\right)^2$$

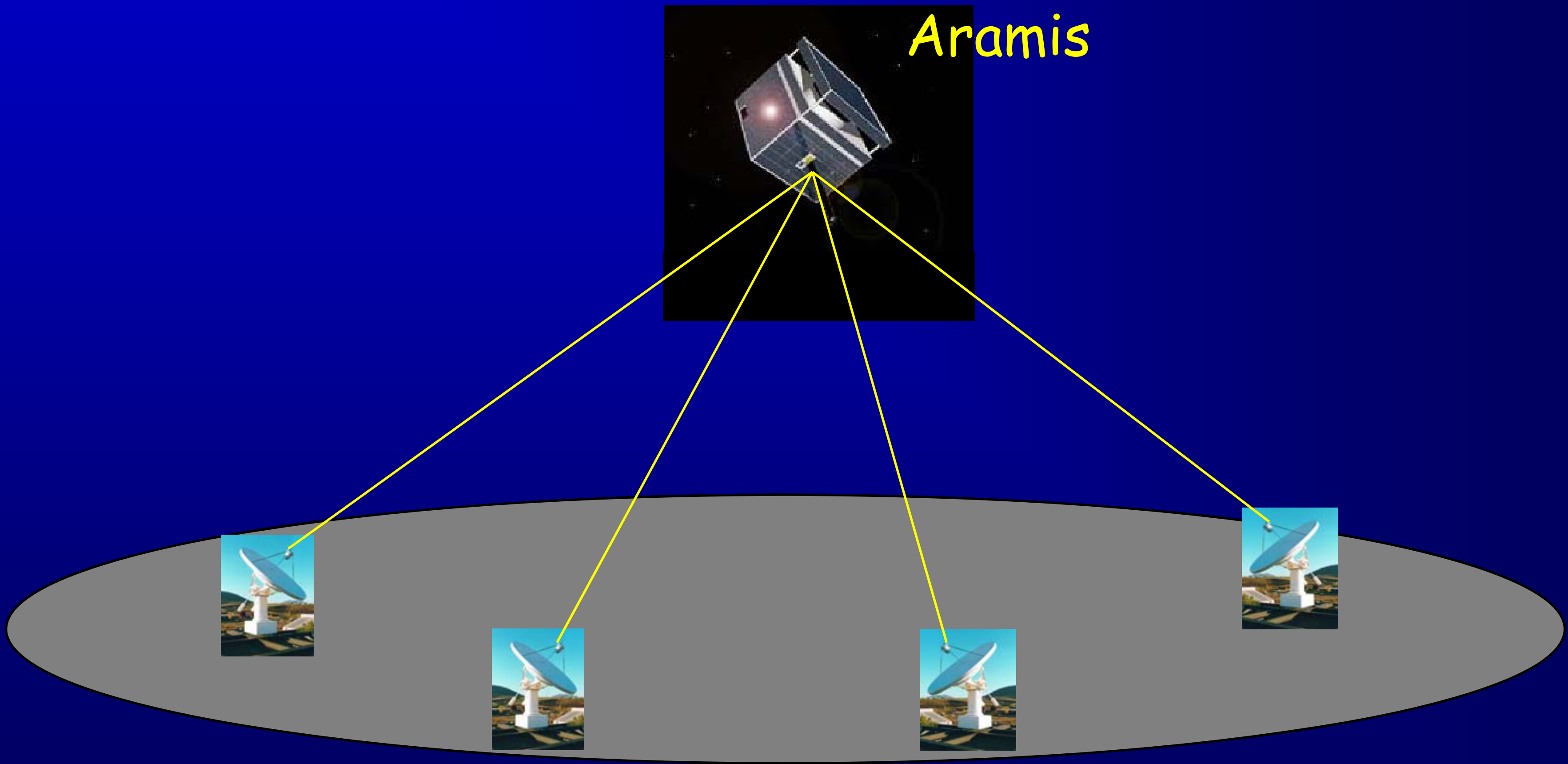
$$A_1 = 2\left(\gamma_b \tau_b - \gamma_a \tau_a\right)$$
$$B_1 = 2\left(x_b - x_a\right)$$
$$C_1 = 2\left(y_b - y_a\right)$$
$$D_1 = 2\left(z_b - z_a\right)$$
$$N_1 = x_a{}^2 - x_b{}^2 + y_a{}^2 - y_b{}^2 + z_a{}^2 - z_b{}^2 - \left(\gamma_a{}^2 \tau_a{}^2 - \gamma_b{}^2 \tau_b{}^2\right)$$

$$\mathcal{D} = \sum_{\not{p}(1,2,3)} sign(\not{p}) B_1 C_2 D_3$$

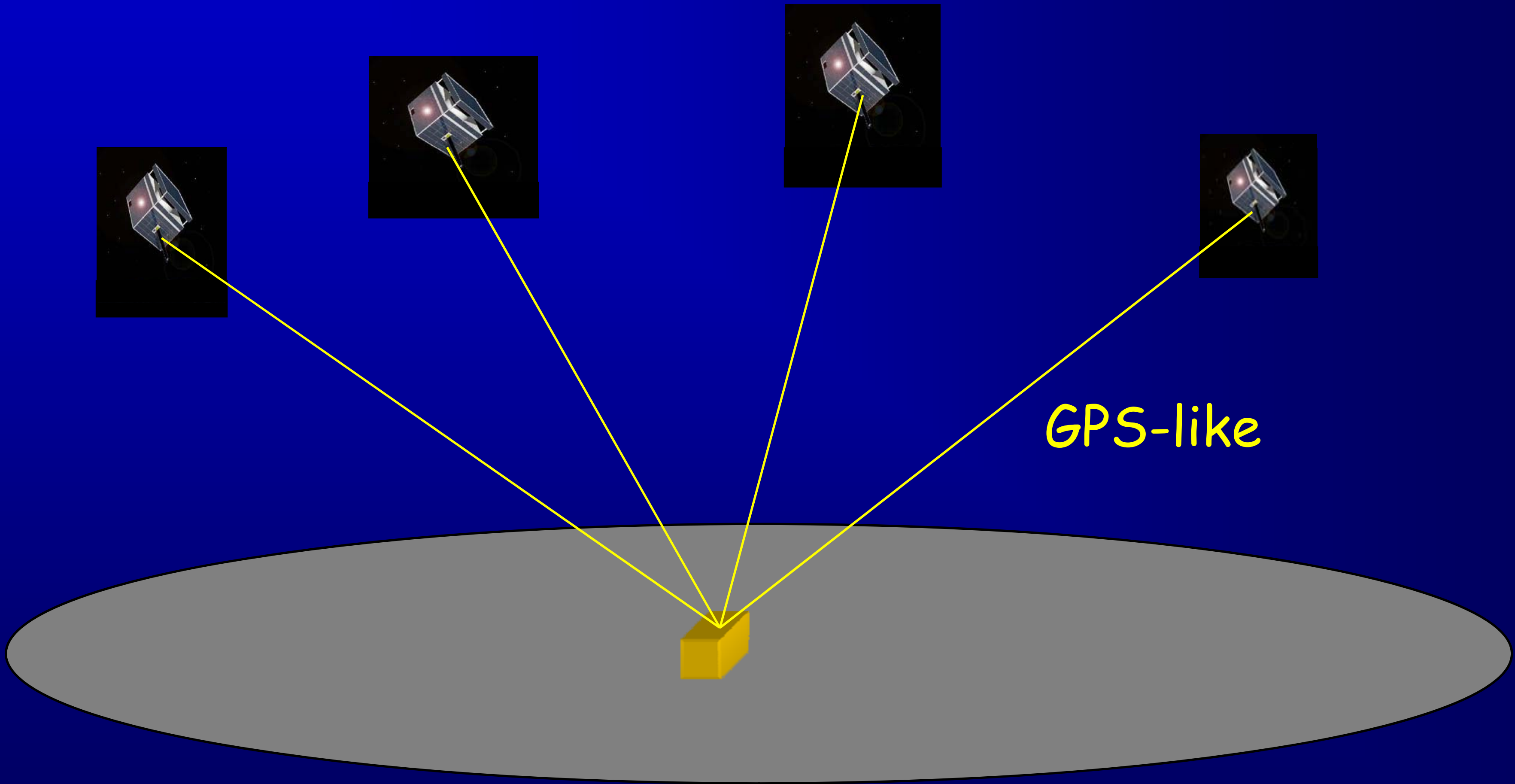
Solutions

$$
\begin{aligned}
T = &-\frac{N_1}{\mathcal{D}}\left(D_2B_3C_4 - D_2B_4C_3 - D_3B_2C_4 + D_3B_4C_2 + D_4B_2C_3 - D_4B_3C_2\right)\\
&+\frac{N_2}{\mathcal{D}}\left(D_1B_3C_4 - D_1B_4C_3 - D_3B_1C_4 + D_3B_4C_1 + D_4B_1C_3 - D_4B_3C_1\right)\\
&-\frac{N_3}{\mathcal{D}}\left(D_1B_2C_4 - D_1B_4C_2 - D_2B_1C_4 + D_2B_4C_1 + D_4B_1C_2 - D_4B_2C_1\right)\\
&+\frac{N_4}{\mathcal{D}}\left(D_1B_2C_3 - D_1B_3C_2 - D_2B_1C_3 + D_2B_3C_1 + D_3B_1C_2 - D_3B_2C_1\right)
\end{aligned}
$$

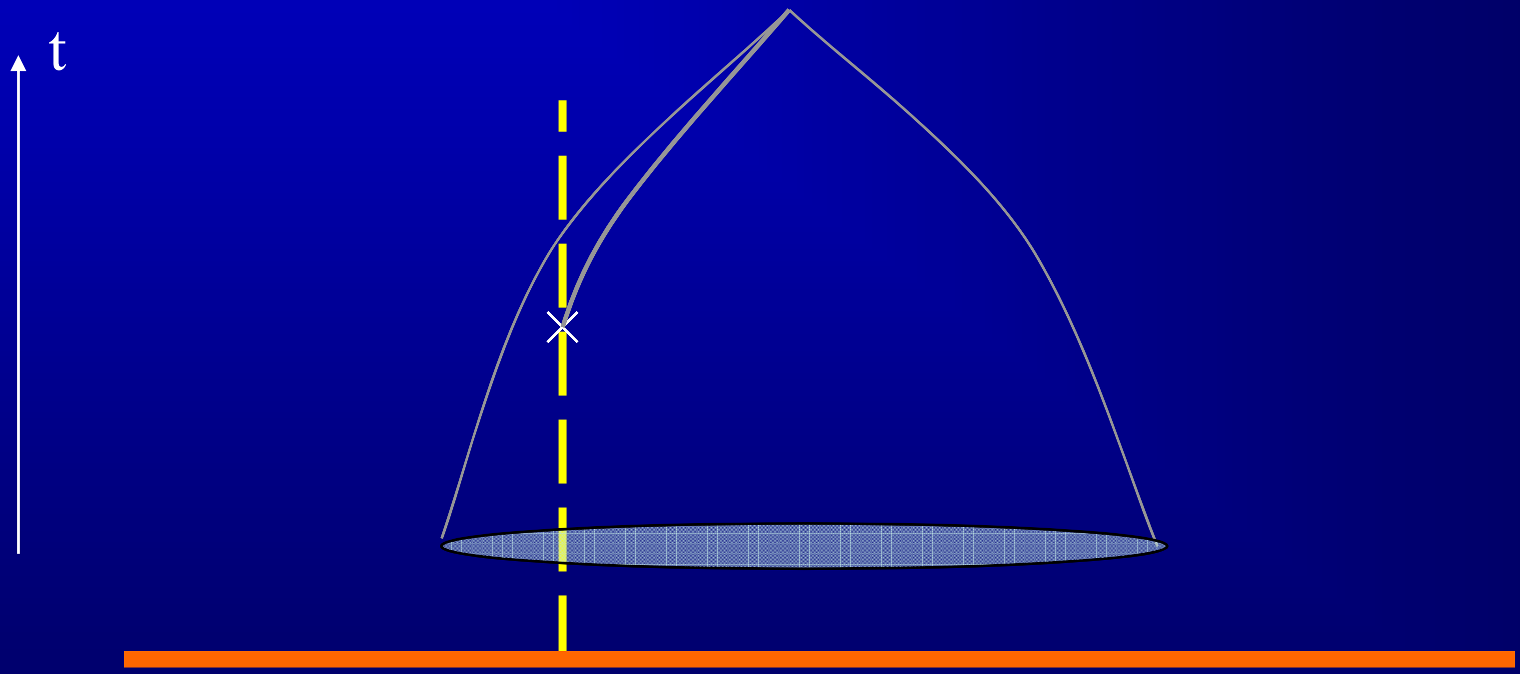
Configurations (a)



Configurations (b)



Gravitational field



Geodesic motion of the clocks

If the motion is *geodesic*:

- Either the equation is known
- Or the number of parameters is known and contains the information about the gravitational field

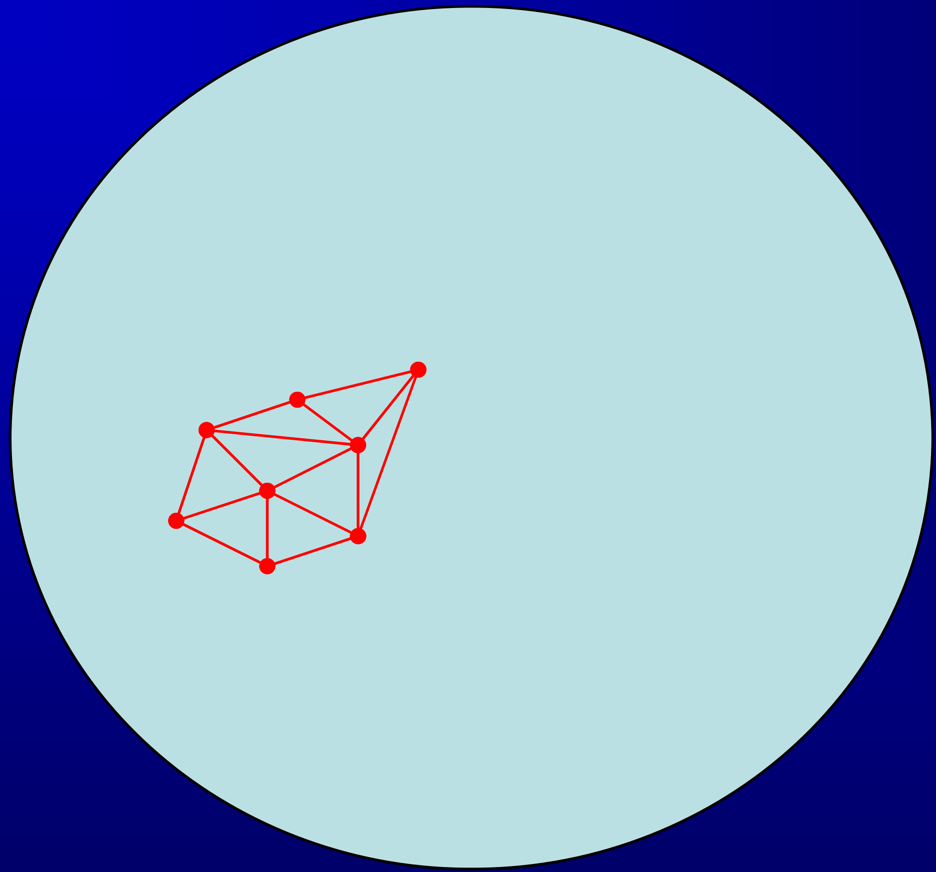
The inverse problem

- Arbitrarily choose 4 fixed points four your fiducial reference frame
- Attribute to them the coordinates values you prefer
- Record the arrival times of the signals from the clocks, at the fiducial points for a long enough interval
- Deduce the parameters of the worldlines of the clocks

Exploring the gravitational field

- Build geodesic triangles
- Cover a region of space time with tetrahedra (Simplexes of the Regge calculus)
- Analyze the geometrical relations in the covering

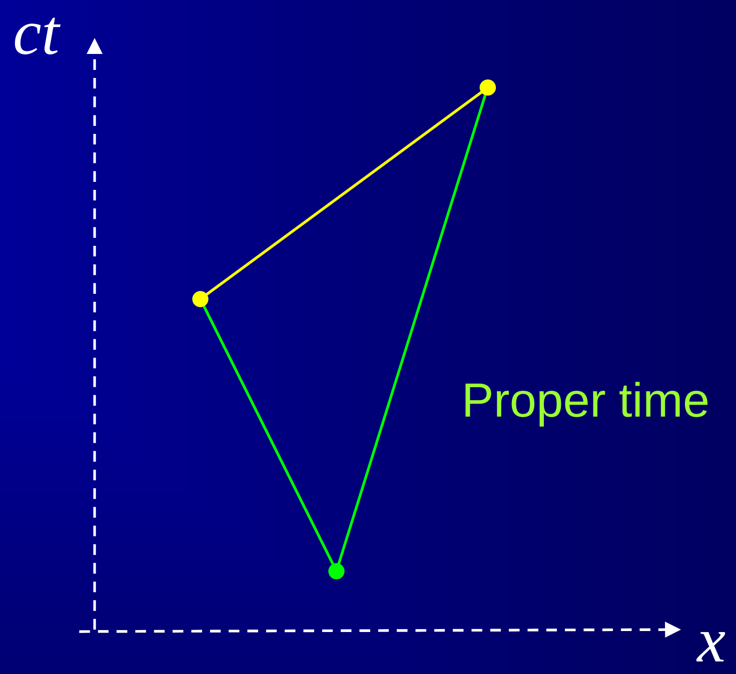
Gaussian mapping of a curved manifold



Geodesic triangles in space-time

Null edge: electromagnetic signal

Time-like edges: freely falling massive test particles



Null tetrads

Let an orthonormal tetrad $e_\alpha \equiv (e_0, e_1, e_2, e_3)$

Build a non-orthogonal null new one:

$$f_0 = \left(\sqrt{3}e_0 + e_1 + e_2 + e_3\right)/2$$
$$f_1 = \left(\sqrt{3}e_0 + e_1 - e_2 - e_3\right)/2$$
$$f_2 = \left(\sqrt{3}e_0 - e_1 + e_2 - e_3\right)/2$$
$$f_3 = \left(\sqrt{3}e_0 - e_1 - e_2 + e_3\right)/2$$

We may write

$$f_\mu = e_\alpha F_\mu{}^\alpha$$

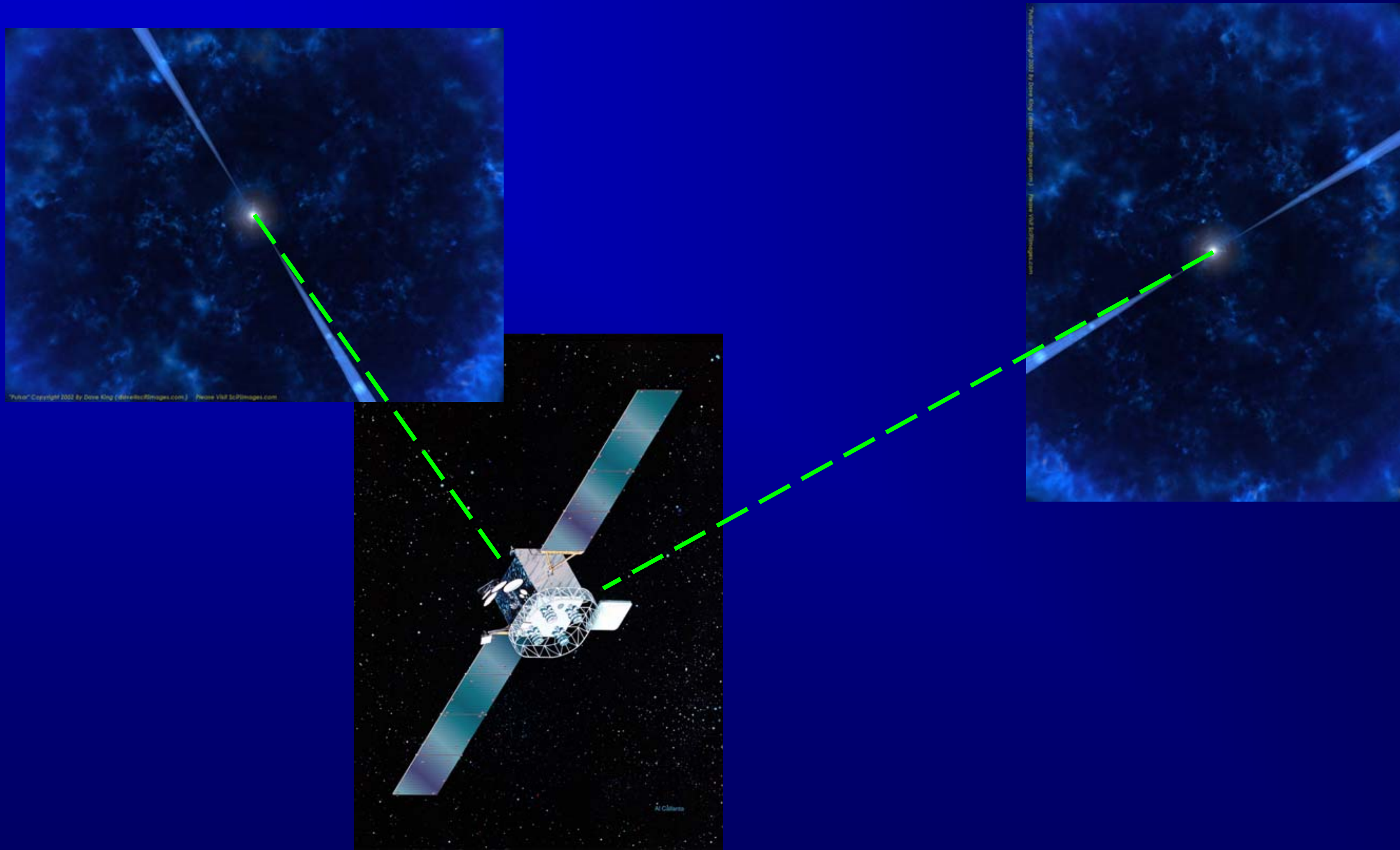
with

$$F_\mu{}^\alpha = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ \sqrt{3} & 1 & -1 & -1 \\ \sqrt{3} & -1 & 1 & -1 \\ \sqrt{3} & -1 & -1 & 1 \end{pmatrix}$$

Null coordinates: the metric tensor (tangent space)

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Space navigation with pulsars

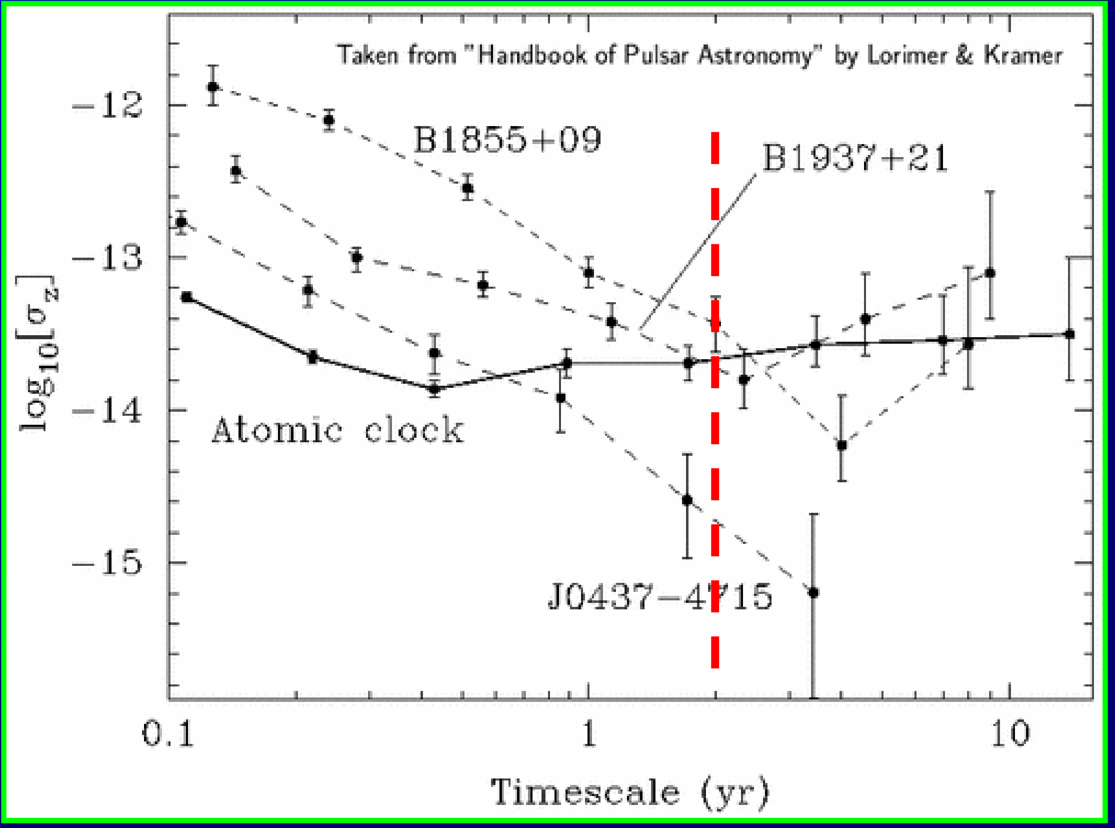


Two options

- X-ray pulsars
- Radio-pulsars

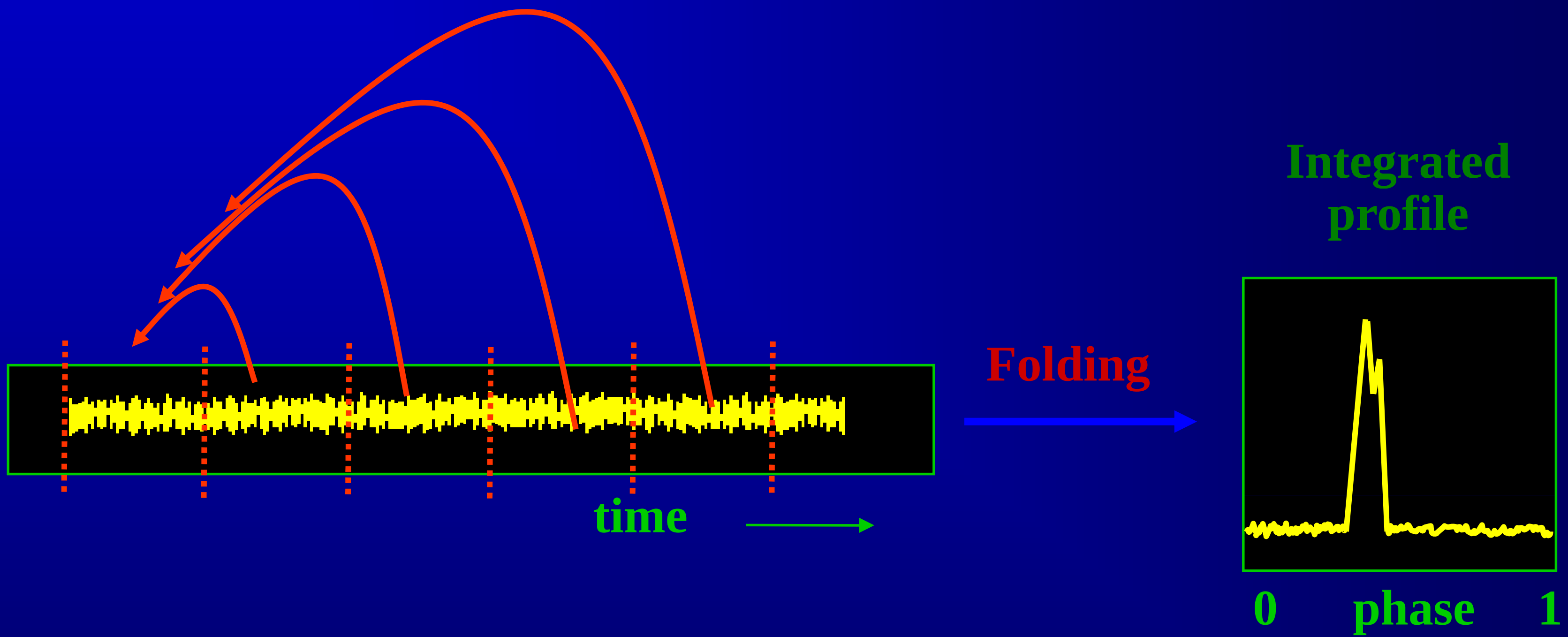
Our choice is millisecond radio-pulsars

Pulsars vs atomic clocks



- ~ 1800 "clocks"
- "Fixed" positions in the sky
- Very stable clocks
- Periods ≥ 1 ms
- Extremely weak signals
- Long integration times

The folding

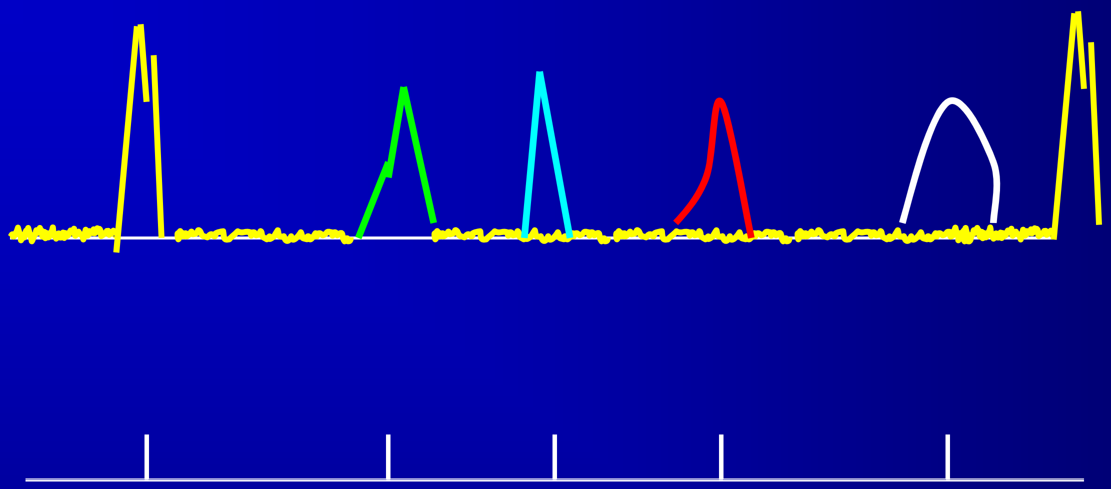


Experiment: Eppur si muove

A modern version of Galileo's sentence

Tracking the Earth's motion with respect to the pulsars

- Data from Parkes
- Collaboration with INAF - Cagliari



The information is contained in the relative phases of the signals from different pulsars

Summary

- In General Relativity gravity is reduced to geometry
- Since electromagnetic signals span null intervals they are particularly fit for building space-time based reference systems
- The traditional techniques of topography can be extended to four-dimensional Riemannian manifolds
- The basic facets are timelike-null-null triangles where "distances" are measured by proper time intervals of freely falling clocks
- As clocks pulsars may be chosen